Games of Incomplete Information
Played By Statisticians

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Abstract

This paper proposes a foundation for heterogeneous beliefs in games, in which disagreement arises not from different information, but from different ways of learning from the same data. A key assumption imposes that while players may interpret the data in different ways, they have common knowledge of a class of procedures for learning from data. Using this framework, I construct a quantitative metric for the analyst’s “confidence” in a strategic prediction, based on the probability that the prediction is supported by beliefs consistent with the realized data. This level of confidence depends on various primitives of the learning environment, including the quantity of data that players observe and the complexity of the learning problem. The main results describe asymptotic confidence as the quantity of data gets large, and provide bounds on confidence when players have observed only a small quantity of data. I show how the proposed approach generates new comparative statics—for example, that speculative trade between agents is more plausible when agents learn from higher-dimensional data, and that coordination is less likely when agents observe noisy data.

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1 Introduction

Predictions of play in incomplete information games depend crucially on the beliefs of the agents, but we rarely know what those beliefs are. The standard approach to modeling beliefs assumes that players share a common prior belief over states of the world, and form posterior beliefs using Bayesian updating. Under this approach, posterior beliefs that are commonly known must be identical (Aumann, 1976), and repeated communication of beliefs eventually leads to agreement (Geanakoplos and Polemarchakis, 1982). These implications conflict not only with considerable empirical evidence of public and persistent disagreement, but also with the more basic, day-to-day, experience that individuals interpret the same information in different ways.  

This paper generalizes from the assumption of a common prior by supposing that players form beliefs based on a set of ways of interpreting the data. Formally, I define a learning rule to be any function that maps data (a sequence of signals) into a belief distribution over payoff-relevant parameters (a first-order belief). Players have common knowledge of some set of “reasonable” learning rules—for example, these learning rules may correspond to Bayesian updating from a set of prior beliefs, or they may correspond to different frequentist estimators for the unknown parameter. The special case of a singleton Bayesian learning rule returns the common prior assumption, but in general, the set of learning rules will infer different parameter values from the same data. This produces a set of plausible parameter values. I impose a key restriction on beliefs to structure the approach: for any realization of the data, players have common certainty in the set of plausible parameter values—that is, they assign probability 1 to the true parameter belonging to this set, suppose that others do as well, and so forth.

The main contribution of the paper is a proposed metric for the analyst’s “confidence” in a strategic prediction in this game. Specifically, consider the prediction that a given action is rationalizable. Whether this is consistent with the proposed belief restriction depends on the realization of the data. I associate each action with a confidence set based on the

\footnote{In financial markets, players publicly disagree in their interpretations of earnings announcements Kandel and Pearson (1995), valuations of financial assets Carlin et al. (2013), forecasts for inflation Mankiw et al. (2004), forecasts for stock movements Yu (2011), and forecasts for mortgage loan prepayment speeds Carlin et al. (2014). players publicly disagree also in matters of politics Wiegel (2009) and climate change Marlon et al. (2013).}
measure of data sets given which that prediction holds. The upper bound of the confidence set is the probability that the action is rationalizable given some belief (hierarchy) satisfying the belief restriction, and the lower bound is the probability that the prediction holds for all beliefs satisfying the restriction. Thus, if both of these probabilities are equal to one, the analyst has maximal certainty that the action is rationalizable, and if they are both zero, he has maximal certainty that it is not. In the intermediate cases, there is uncertainty about whether the action is rationalizable, and the confidence sets reflect the extent of that uncertainty.

The main results in this paper characterize various properties of these confidence sets, beginning with their asymptotic behaviors. I first show that if sets of learning rules are too large, then the confidence set may fail to be asymptotically continuous: that is, even if an action is strictly rationalizable in the limiting game, the analyst’s confidence sets may be very different from \( \{1\} \) for arbitrarily large quantities of data. Roughly, this is because the rate of convergence under different learning rules cannot be uniformly bounded, so it is always possible that some learning rule produces an expected payoff that is very different from the limiting value. If, however, the set of learning rules satisfy a uniform convergence property that I describe, then the following statements hold: If an action is strictly rationalizable at the limit, then the analyst’s confidence set must converge to \( \{1\} \) as the quantity of data gets large, and if an action is not rationalizable in the limit, the analyst’s confidence set converges to \( \{0\} \). (The intermediate case, in which actions are rationalizable but not strictly rationalizable, is more subtle—see Section 5 for further detail.)

Next, I consider the setting of small sample sizes, and bound the extent to which the analyst’s confidence set differs from its asymptotic limit. The main results in this section provide quantitative bounds for the confidence sets, which connect the analyst’s confidence to primitives of the learning environment. For example, if an action is strictly rationalizable in the limiting game, then whether it is rationalizable for finite amounts of data depends on how fast the different learning rules jointly recover the payoff-relevant parameter, as well as on a cardinal measure for how strict the solution is at the limit. These results complement the examples in Sections 2 and 4, where I explicitly characterize confidence sets in two classic settings—a trade game and a coordination game—and show how the proposed approach can be used to generate new comparative statics. For example: speculative trade is more plausible when agents learn from higher-dimensional data, and that coordination is less likely
when agents observe noisy data. These predictions—which hold even if data is public and common—are difficult to produce under assumption of a common prior.

This paper contributes to a large literature studying the robustness of strategic predictions to the specification of player beliefs (Rubinstein, 1989; Dekel et al., 2006; Weinstein and Yildiz, 2007; Chen et al., 2010) and equilibrium selection in incomplete information games (Carlsson and van Damme, 1993; Kajii and Morris, 1997). At a technical level, the restrictions on beliefs that I place ensure that permitted types converge in the uniform-weak topology, as proposed and characterized in Chen et al. (2010) and Chen et al. (2017). This relationship is described in more detail in Section 5.2.

Conceptually, the goals of the present paper differ from the previous literature in several respects: First, my focus here is not on equilibrium selection—choosing one equilibrium from a set of many—but rather on providing a metric for the strength of a given prediction. Second, in contrast to the many binary or “qualitative” notions of robustness that have been proposed, this paper delivers a continuous-valued or quantitative metric.

Third, while the literature thus far has primarily considered robustness to perturbations of beliefs, I am interested here also in predictions that we may make for beliefs that are “far away” from the limiting beliefs. To discipline these beliefs, I endogenize the type space using a learning foundation for belief formation. In strategic contexts, such learning-based approaches have been explored in Dekel et al. (2004) and Esponda (2013) among others, although my goal here is to provide a metric of robustness rather than a new solution concept. Other important precedents include Cripps et al. (2008) and Acemoglu et al. (2015), which study how beliefs (about beliefs) evolve given increasing quantities of data, and Steiner and Stewart (2008), which characterizes the limiting equilibria of a sequence of games in which players infer payoffs from related games. Finally, the depiction of agents as “statisticians” or “machine learners” relates to a growing literature in decision theory (Gilboa and Schmeidler, 2003; Gayer et al., 2007; Al-Najjar, 2009; Al-Najjar and Pai, 2014) and game theory (Jehiel, 2005; Spiegler, 2016; Olea et al., 2017; Cherry and Salant, 2020). In particular, Cherry and Salant (2020) similarly models players in a game as statisticians who form beliefs based on data, although in that paper, players use the same rule to interpret different data, while I focus on players who use different rules to interpret common data.
2 Example

In this section, I use the proposed approach to revisit a classic example: trade between two players. In this game, the assumption that players share identical posterior beliefs if they see common data has strong implications for strategic play. I show how we can relax this assumption by endogenizing disagreement based on two new primitives—a data-generating process and set of learning rules—and relate “confidence” in prediction of trade back to primitives of the learning environment.

The game is described as follows: A Seller owns a good of unknown value $v \in \{0, 1\}$. He can either enter a market at cost $c$, or exit and keep the good. Entering leads to a simultaneous interaction with a Buyer, where the Seller chooses whether to sell the good at a (pre-set) posted price $p$, and the Buyer chooses whether to purchase the good at that price. The game and its payoffs are described in Figure 1 below.

![Figure 1: Description of Game](image)

Suppose that the cost $c$ and price $p$ satisfy $0 < c < p < 1$, so the seller prefers to sell at the low value and prefers to keep the good at the high value. If players hold identical posterior beliefs, then entering is not rationalizable for the Seller in this game, reminiscent of the no-trade theorem (Milgrom and Stokey, 1982).\(^2\)

I relax this assumption by supposing that players commonly observe a sequence of past goods and valuations, where each good is described by a vector $x \in X := [-1, 1]^m$ of $m$ \(^2\)

\(^2\)If trade does not occur subsequently, then the Seller receives $v - c$ from entering but $v > v - c$ from exiting. Thus, entering can be rationalized only if trade subsequently occurs. But trade can occur only if the Buyer believes that $\mathbb{E}(v) \geq p$ while the Seller believes that $\mathbb{E}(v) \leq p$, implying $\mathbb{E}(v) = p$ under their shared belief. The Seller can improve on his expected payoff of $p - c$ by choosing to exit.
observable attributes normalized to lie in the interval $[-1, 1]$. A good with attributes $x$ has value $f(x)$, where $f$ is an unknown mapping. Players commonly observe a data set $z_n = \{(x_i, f(x_i))\}_{i=1}^n$ consisting of $n$ goods $x_i$ drawn from a uniform distribution on $X$, along with their values $f(x_i)$.

Although agents do not know $f$, they do know that $f$ belongs to a certain family of functions. For simplicity, take this to be the set $\mathcal{F}$ of rectangular classification rules, i.e. functions $f_R(x) = 1(x \in R)$ indexed to hyper-rectangles $R$ in $[-1, 1]^m$. A Bayesian statistician with prior $\pi \in \Delta(\mathcal{F})$ over rectangular classification rules re-normalizes his prior over all classification rules consistent with the observed data—that is, all $\tilde{f} \in \mathcal{F}$ such that $\tilde{f}(x_i) = f(x_i)$ at the observed $x_i$. (See Figure 2 for examples.) Suppose the properties of the Seller’s good are known, and for simplicity let it be the zero-vector, denoted $x_0$. The Bayesian’s expectation of $v = f(x_0)$ is then given by $\mathbb{E}_\pi(v | z) := \int_{f \in \mathcal{F}} f(x_0) \pi(f | z) df$.

![Figure 2](image)

**Figure 2:** The circles represent the observed data. Each good is described by a vector in $[-1, 1] \times [-1, 1]$. The circle is black if its valuation is 1 and gray if its valuation is low. A rule is consistent with the data if it correctly predicts the valuation for each observation. Two rectangular classification rules are depicted: each predicts ‘1’ for goods in the shaded region and ‘0’ for goods outside. Both are consistent with the observed data.

Without restricting which priors are valid, we can discipline beliefs with the following weaker requirement: players assign probability 1 to the set of expectations of $v$ that are consistent with Bayesian updating from some prior over rectangular classification rules, assign

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3For example, whether all attributes fall into an “acceptable” range, as judged by a downstream buyer.
probability 1 to the other player assign probability 1 to this set, and so on.

**Assumption 1** (Restriction on Beliefs). For every $z_n$, players have common certainty in the set $\mathcal{C}(z_n) = \{E_\pi(v \mid z_n) : \pi \in \Delta(F)\}$.

Since the set $\mathcal{C}(z_n)$ is endogenous to the data, the set of beliefs satisfying Assumption 1 is also random from the ex-ante perspective of the analyst. We can thus quantify the analyst’s “confidence” in predicting that entering is rationalizable for the Seller as follows: Let $\mathcal{P}$ be the measure of data sets $z_n$ such that entering is rationalizable for some belief satisfying Assumption 1, and let $\mathcal{P}^\prime$ be the measure of data sets $z_n$ such that entering is rationalizable for all beliefs satisfying Assumption 1. Then the set $[\mathcal{P}, \mathcal{P}^\prime]$ is a kind of “confidence set” that describes how certain the analyst should be in predicting that the entering is rationalizable, when agents have commonly observed $n$ data points. The extreme case of $\mathcal{P}^\prime = \mathcal{P} = 1$ implies full confidence that entering is rationalizable, while $\mathcal{P}^\prime = \mathcal{P} = 0$ implies full confidence that it is not.

**Claim 1.** The probability $\mathcal{P} = 0$ for every $n$. Additionally:

(a) Fixing any number of attributes $m$, the probability $\mathcal{P}^\prime \to 0$ as $n \to \infty$.

(b) Fixing any number of data observations $n$, the probability $\mathcal{P}^\prime$ is increasing in the number of attributes $m$, and $\mathcal{P}^\prime \to 1$ as $m \to \infty$.

That is, for every quantity of data $n$, the probability that entering is rationalizable for all beliefs satisfying Assumption 1 is zero. But the probability that entering is rationalizable for some belief satisfying Assumption 1 varies depending on $n$ and $m$. Part (a) says that as the number of observations $n$ grows large, this probability $\mathcal{P}^\prime$ vanishes to zero, implying that the confidence set converges to a degenerate interval at zero. The infinite-data limit thus returns the prediction of “no trade” consistent with the common prior assumption. But if the amount of data is finite, and the number of attributes is large, then $\mathcal{P}^\prime$ can be substantially greater than zero. Indeed, Part (b) of the claim says that this probability $\mathcal{P}^\prime$ can be made arbitrarily close to 1 by increasing the dimensionality of the learning problem via choice of large $m$. This reflects that in a high-dimensional learning problem, many classification rules are likely to be consistent with the data, including some that yield conflicting predictions. Thus, “rational” disagreement is possible and even likely.
An exact characterization of \([\underline{p}^n, \bar{p}^n]\) is given in Lemma 3 in the appendix. Using these expressions, I plot below the behavior of these confidence sets for different numbers of attributes \(m\), taking the true function to be \(1(x \in [-a, a]^m)\) where \(a = 0.1\).

Figure 3: The shaded area depicts confidence sets \([\underline{p}^n, \bar{p}^n]\) for the rationalizability of entering given \(n\) common observations.

Thus, although trade is not predicted in the limiting game, it is a plausible outcome if the number of data observations is small and the number of attributes is large. For example, if there are 10 attributes and players observe only 20 goods, then the confidence set \([\underline{p}^n, \bar{p}^n]\) = [0, 0.99]. That is, with near certainty, entering will be rationalizable for the Seller given the realized data for some belief satisfying Assumption 1.

A first takeaway from this exercise is qualitative: Claim 1 predicts that actions which require disagreement in beliefs are more likely to occur when those beliefs are based on data in “complex” learning environments. While this is an intuitively reasonable prediction, it is difficult to obtain under the common prior assumption, which not only precludes different beliefs given identical data, but also precludes disagreement given different data, if posterior beliefs are common knowledge (Aumann, 1976).

The second takeaway is quantitative: computations like those used to produce Figure 3 can provide a sense of how much data agents need to see to justify a given strategic prediction. This exercise does require the analyst to have a model of what learning rules agents consider reasonable in the setting, and what the true data-generating process is. While these are not small demands, these new primitives are more observable than agent beliefs in many economic environments.\(^4\) Moreover, while I chose a simple set of learning

\(^4\)For example, the set of statistical procedures that are commonly used for interpreting certain data sets
rules for the purpose of obtaining exact expressions for the confidence set, in practice such
confidence sets can be simulated for more complex kinds of learning procedures.

Subsequently, I generalize the approach described in this example.

3 Approach

3.1 Preliminaries

Basic Game. There is a finite set of $I$ players and a finite set of actions $A_i$ for each player $i$. The set of action profiles is $A = \times_i A_i$, and the set of possible games is identified with $U := \mathbb{R}^{|I| \times |A|}$. Agents have beliefs over a set of payoff-relevant parameters $\Theta$, which is a compact subset of finite-dimensional Euclidean space. It is possible to set $\Theta = U$, so that $\Theta$ is itself the set of games, or to define beliefs over a lower-dimensional set of payoff-relevant parameters, as in the previous example. In either case, the parameters in $\Theta$ are assumed to be related to payoffs by a bounded and Lipschitz continuous embedding $g : \Theta \to U$ (assuming the sup-norm on both spaces).\(^5\) When convenient, the notation $u_i(a, \theta) = g_i(\theta)$ is used to denote player $i$’s payoff given action profile $a$ and parameter $\theta$.

Beliefs. Let $X_0 = \Theta$, $X_1 = X_0 \times (\Delta(X_0))^f$, $X_2 = X_1 \times (\Delta(X_1))^f$, etc., so that each $X_k$ is the set of possible $k$-th order beliefs. Define $T_0 = \prod_{n=0}^\infty \Delta(X_n)$. An element $(t_1^i, t_2^i, \ldots) \in T_0$ is a hierarchy of beliefs over $\Theta$ (describing the player’s uncertainty over $\Theta$, his uncertainty over his opponents’ uncertainty over $\Theta$, and so forth), and referred to simply as a belief or type. There is a subset of types $T_i^* \subseteq T_i$ (that satisfy the property of coherency\(^6\) and common knowledge of coherency) and a function $\kappa_i^* : T_i^* \to \Delta(\Theta \times T_i^*)$ such that $\kappa_i^*(t_i^*)$ preserves the beliefs in $t_i$; that is, $\text{marg}_{X_{n-1}} \kappa_i^*(t_i^*) = t_i^n$ for every $n$ (Mertens and Zamir, 1985; Brandenburger and Dekel, 1993).\(^7\) The tuple $(T_i^*, \kappa_i^*)_{i \in \mathcal{I}}$ is the universal type space. In practice, modelers often work with smaller (belief-closed) type spaces $(T_i, \kappa_i)_{i \in \mathcal{I}}$ where each $T_i \subseteq T_i^*$ and $\kappa_i : T_i \to \kappa_i^*(T_i)$ is the restriction of $\kappa_i^*$ to $T_i$.\(^8\)

\(^5\)The map $g$ can be interpreted as capturing the known information about the structure of payoffs.

\(^6\)may be commonly understood as part of standard industry practice. In some industries, a certain set of statistical frameworks are taught to professionals as part of their training.

\(^7\)The map $g$ can be interpreted as capturing the known information about the structure of payoffs.

\(^8\)marg$_{X_{n-2}} t_i^n = t_i^{n-1}$, so that $(t_1^i, t_2^i, \ldots)$ is a consistent stochastic process.

\(^9\)Notice that $T_i^*$ is used here to denote the set of profiles of opponent types.
Types are sometimes modeled as encompassing all uncertainty in the game. In the present paper, types describe players’ structural uncertainty over payoffs, but not their strategic uncertainty over opponent actions. The set of states of the world is \( \Omega = \Theta \times \prod_{i \in I} T_i^s \), where each \( \omega \in \Omega \) describes the resolution of the payoff-relevant state as well as of all player types.

### 3.2 Restriction on Beliefs

The proposed approach endogenizes the type space based on two new primitives: a data-generating process, and a set of rules for how to extrapolate beliefs from realized data. Formally, let \((Z_t)_{t \in \mathbb{Z}}\) be a stochastic process where the random variables \(Z_t\) take value in a common set \(\mathcal{Z}\), and the typical sample path is denoted \(z = (z_1, z_2, \ldots)\). The data-generating process is a measure \(P\) over the set \(\mathcal{Z}^\infty\) of all (infinite) sample paths. Let \(P^n\) denote the induced measure on the first \(n\) variables. A data set \(z_n\) of size \(n\) is the restriction of \(z\) to its first \(n\) coordinates, and \(\mathcal{Z}^n\) is the set of all length-\(n\) data sets. In a Bayesian formulation, we would write \(P_\theta \in \Delta(\mathcal{Z}^\infty)\) for the data-generating process associated with parameter \(\theta\), and assume that \((P_\theta)_{\theta \in \Theta}\) is common knowledge. The likelihoods \((P_\theta)_{\theta \in \Theta}\) along with a distribution over \(\Theta\) complete the description of a prior over \(\Theta \times \mathcal{Z}^\infty\), and if this prior is shared across agents, then we return the common prior approach. The realized data \(z_n\) uniquely determines a posterior belief over \(\Theta\).

Generalizing from Bayesian updating, I define a learning rule to be any map from data sets into first-order beliefs:

\[
\mu : \bigcup_{n=1}^{\infty} \mathcal{Z}^n \to \Delta(\Theta).
\]

For example, a learning rule may map the history \(z_n\) to a degenerate belief at a sample statistic (such as the empirical average\(^8\)) or to a distribution over various point-estimates for \(\theta\). Throughout this paper, I restrict consideration to learning rules that satisfy the following condition:

**Assumption 2** (Common Limiting Parameter). *There is a limiting parameter \(\theta^\infty\) such that*

\[
\lim_{n \to \infty} \mathbb{E}_{\mu(z_n)}(\theta) \to \theta^\infty \quad P\text{-a.s.}
\]

This assumption requires that each learning rule recovers the same limiting parameter \(\theta^\infty\) as the quantity of data \(n\) grows large. If we interpret \(\theta^\infty\) to be the “true” value of the

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\(^8\)See Jehiel (2018) for a recent paper that studies agents who learn from data using such a rule.
parameter, then this assumption implies that each \( \mathbb{E}_{\mu(z_n)}(\theta) \) is a consistent estimator of \( \theta^\infty \).

Players have common knowledge of a set \( \mathcal{M} \) of learning rules satisfying the above assumption, where these rules may interpret data in different ways. The set of “plausible” expected parameters given this set \( \mathcal{M} \) and the realized data \( z_n \) is

\[
\mathcal{C}(z_n) = \{ \mathbb{E}_{\mu(z_n)}(\theta) \mid \mu \in \mathcal{M} \}
\]

(1)

I assume that players have common certainty in this set \( \mathcal{C}(z_n) \)—that is, they assign probability 1 to the set, believe with probability 1 that all other players assign probability 1 to the set, and so forth (Monderer and Samet, 1989).

Formally, for any set \( \mathcal{E} \subseteq \Theta \), define \( B_i^{1,1}(\mathcal{E}) := \{ t_i \in T_i^* : \text{marg}_{\Theta} \kappa_i^*(t_i)[\mathcal{E}] = 1 \} \) to be the set of player \( i \) types whose marginal beliefs over \( \Theta \) assign probability 1 to the set \( \mathcal{E} \). Recursively, for each \( k > 1 \), define \( B_i^{k,1}(\mathcal{E}) = \{ t_i \in T_i^* : \kappa_i^*(t_i) \left( \Theta \times B_{-i}^{k-1,1} \right) = 1 \} \). Then

\[
T_i^{\mathcal{E}} = \bigcap_{k \geq 1} B_i^{k,1}(\mathcal{E})
\]

is the set of player \( i \) types that have common certainty in \( \mathcal{E} \times \prod_{i \in I} T_i^* \), or more simply, common certainty in \( \mathcal{E} \).

The (interim) type space given data \( z_n \) is defined as follows:

**Definition 1.** For every \( z_n \), the induced type space is \( (T_i^{\mathcal{E}(z_n)}, \kappa_i^{\mathcal{E}(z_n)}) \), where \( \kappa_i^{\mathcal{E}} : T_i^{\mathcal{E}} \to \kappa_i^*(T_i^{\mathcal{E}}) \) is the restriction of \( \kappa_i^* \) to \( T_i^{\mathcal{E}} \). Say that the type \( t_i \) is permitted for player \( i \) if \( t_i \in T_i^{\mathcal{E}(z_n)} \).

This type space includes all type profiles where each player \( i \) has common certainty in the set of parameters \( \mathcal{C}(z_n) \). Note that the type space permits common knowledge disagreement—that is, player \( i \) can believe with probability 1 that (all believe with probability 1 that...) players hold different first-order beliefs. Such types are precluded under the common prior assumption not only in the present setting of common data, but also if we were to allow

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\(^9\)This assumption can be relaxed, see Section 7.

\(^{10}\)As I discuss in Section 7, it is not critical that players have common certainty in the set, and the assumption can be relaxed to common \( p \)-belief in \( \mathcal{C}(z_n) \) for large \( p \).

\(^{11}\)For example, \( B_i^{2,1}(\mathcal{E}) \) is the set of player \( i \) types that assign probability 1 to all other players assigning probability 1 to \( \mathcal{E} \).

\(^{12}\)It is easy to show that the type spaces \( (T_i^{\mathcal{E}}, \kappa_i^{\mathcal{E}})_{i \in I} \) are belief-closed; that is, \( \kappa_i^{\mathcal{E}}(t_i) (\Theta \times T_{-i}^{\mathcal{E}}) = 1 \) for every \( t_i \in T_i^{\mathcal{E}} \).
for private and different information (Aumann, 1976). The induced type spaces in Definition 1 thus corresponds to a relaxation of the common prior assumption, where the permitted extent of disagreement is governed by the set of learning rules $\mathcal{M}$.

Note also that restrictions are placed only on the final beliefs that players hold, and not on how they came to form those beliefs. In particular, I do not impose a structural model for how the players use learning rules from $\mathcal{M}$. For example, each of the following is consistent with the restriction in Definition 1:

- **Uncertainty**: Each player has a uniform belief over $\mathcal{C}(z_n)$.

- **Randomization**: All players form first-order beliefs by drawing a learning rule $\mu$ at random from a public distribution over $\mathcal{M}$ and taking an expectation of $\theta$ with respect to $\mu(z_n)$.

- **Misspecification**: Player $i$ believes that player $j$’s first-order belief is degenerate on $\mathbb{E}_{\mu(z_n)}(\theta)$ for some learning rule $\mu \in \mathcal{M}$, while it is degenerate on $\mathbb{E}_{\mu'(z_n)}(\theta)$ for some other $\mu' \in \mathcal{M}$.

In some cases, such as in the subsequent example in Section 4, it is possible to take $\mathcal{C}(z_n)$ itself as a primitive without explicitly defining $\mathcal{M}$. Other example classes of learning rules include:

**Bayesian Updating with Different Priors.** Let each learning rule $\mu_n \in \mathcal{M}$ be identified with a prior distribution $\pi \in \Delta(\Theta \times \mathcal{Z}^\infty)$. Then for any $z_n$, the belief $\mu_n(z_n)$ is the marginal over $\Theta$ of the posterior belief associated with the corresponding prior $\pi$.

**Sample Statistics.** The set $\mathcal{M}$ consists of learning rules that map the data to different point-estimates for the payoff-relevant parameter. For example, $\mathcal{M}$ might consist of the two learning rules $\mu_{\text{mean}}$ and $\mu_{\text{median}}$, where for any data set $z_n$, $\mu_{\text{mean}}(z_n)$ is a point-mass belief on the mean realization in $z_n$, and $\mu_{\text{median}}(z_n)$ is a point-mass belief on the median realization.

**Linear Regression.** Suppose that $X \subseteq \mathbb{R}^p$, $p < \infty$, is a set of attributes that determine the value of a parameter in $\Theta$ (e.g. physical covariates of a patient seeking health insurance, and medical outcomes for those patients). The observations in $z_n$ are pairs $(x, \theta)$, and the payoff-relevant unknown is the parameter associated with some new $x^*$ (e.g. the outcome for a new patient with characteristics $x^*$).
Each learning rule $\mu \in \mathcal{M}$ corresponds to a different regression model based on a subset of attributes $I_\mu \subseteq \{1, \ldots, p\}$. Write $x_\mu = (x_i)_{i \in I_\mu}$ for the coordinates of $x$ at those indices. Then, $f_{\mu, \text{OLS}}(z_n)(x) = \beta_{\mu, \text{OLS}}(x_\mu)$ is the linear function of the attributes in $I_\mu$ that best fits the observed data $z_n = \{(x^k, \theta^k)\}_{k=1}^n$; that is,

$$\beta_{\mu, \text{OLS}} = \arg\min_{\beta \in \mathbb{R}^{|I_\mu|}} \frac{1}{n} \sum_{i=1}^n (\beta \cdot x^i_\mu - \theta^i)^2$$

For every $z_n$, the learning rule $\mu$ maps $z_n$ into a point-mass belief on the corresponding prediction $f_{\mu, \text{OLS}}(z_n)(x^*)$.

**Case-Based Learning with Different Similarity Functions.** As in the previous example, suppose that the observations are pairs $(x, \theta) \in X \times \Theta$, and the payoff-relevant parameter is the outcome at some new $x^*$. Each “case-based” learning rule $\mu \in \mathcal{M}$ is identified with a real number $\lambda \in \mathbb{R}_+$ (to be interpreted momentarily) and maps the historical data $z_n = \{(x^k, \theta^k)\}_{k=1}^n$ into a weighted average of the observed parameter values $\theta^k$ (Gilboa and Schmeidler, 1995; Gilboa et al., 2008). The observed parameters at $x$-values “more similar” to $x^*$ are weighted more heavily. Formally, let $g : X \times X \to \mathbb{R}_+$ be a similarity function on attributes, where $g(x, x')$ describes the distance between attribute vectors $x$ and $x'$. The learning rule with parameter $\lambda$ maps $z_n$ to a point mass on the weighted average

$$\frac{1}{n} \sum_{k=1}^n \frac{\theta^k e^{-\lambda g_\mu(x^k, x^*)}}{\sum_{k'} e^{-\lambda g_\mu(x^{k'}, x^*)}}.$$

The parameter $\lambda$ controls the degree to which similar observations are weighted more heavily than dissimilar observations. For example, $\lambda = 0$ returns a simple average of all of the observed states, while $\lambda \to \infty$ returns the observed state at the most similar attribute vector.

### 3.3 Analyst’s Confidence Set

I now use the proposed framework to construct a quantitative metric for the analyst’s confidence in a strategic prediction, focusing on prediction that an action is *interim-correlated rationalizable* (Dekel et al., 2007). As explained in Section 7, the choice of this particular solution concept is not critical to the approach—for example, we could alternatively
consider prediction of Bayesian Nash equilibria. I don’t consider this as the primary solution concept, because equilibrium notions are known to lead to potentially counterintuitive predictions when players have common knowledge disagreement.\(^\text{13}\)

The definition of (interim-correlated) rationalizability is reviewed here: Fix any type space \((T^\varepsilon_i, \kappa^\varepsilon_i)_{\varepsilon \in \mathcal{I}}\). For every player \(i\) and type \(t_i \in T^\varepsilon_i\), set \(S^0_i[t_i] = A_i\), and define \(S^k_i[t_i]\) for \(k \geq 1\) such that \(a_i \in S^k_i[t_i]\) if and only if \(a_i\) is a best reply to some \(\pi \in \Delta(\Theta \times T^\varepsilon_i \times A_{-i})\) satisfying (1) \(\arg\max_{\Theta \times T^\varepsilon_i} \pi = \kappa^\varepsilon_i(t_i)\) and (2) \(\pi(a_{-i} \in S^{k-1}_{-i}[t_{-i}]) = 1\), where \(S^{k-1}_{-i}[t_{-i}] = \prod_{j \neq i} S^{k-1}_j [t_{-j}]\). We can interpret \(\pi\) to be an extension of type \(t_i\)’s belief \(\kappa^\varepsilon_i(t_i)\) onto the space \(\Delta(\Theta \times T^\varepsilon_i \times A_{-i})\), with support in the set of actions that survive \(k-1\) rounds of iterated elimination of strictly dominated strategies for types in \(T^\varepsilon_i\). For every \(i\), the actions in \(S^\varepsilon_i[t_i] = \bigcap_{k=0}^{\infty} S^k_i[t_i]\) are interim correlated rationalizable for player \(i\) of type \(t_i\), or (henceforth) simply rationalizable.

For any set of parameter values \(\mathcal{C} \subseteq \Theta\), say that action \(a_i\) is strongly \(\mathcal{C}\)-rationalizable if it is rationalizable for player \(i\) of any type \(t_i \in T^\varepsilon_i\), and it is weakly \(\mathcal{C}\)-rationalizable if it is rationalizable for player \(i\) of some type \(t_i \in T^\varepsilon_i\). Strong and weak \(\mathcal{C}\)-rationalizability are new definitions, and represent two edge approaches—maximally stringent and maximally lenient—for determining whether \(a_i\) constitutes a “reasonable” prediction in the interim type space \((T^\varepsilon_i, \kappa^\varepsilon_i)_{\varepsilon \in \mathcal{I}}\).

The main concept of a confidence set is now defined.

**Definition 2.** For every \(n \in \mathbb{Z}_+\), define \(p^n(a_i)\) to be the probability (over possible datasets \(z_n\)) that action \(a_i\) is rationalizable for every type in \(T^\varepsilon_i(z_n)\); that is,

\[
p^n(a_i) = P^n(\{z_n : a_i \text{ is strongly } \mathcal{C}(z_n)\text{-rationalizable}\}). \tag{2}
\]

Define \(\overline{p}^n(a_i)\) to be the probability (over possible datasets \(z_n\)) that action \(a_i\) is rationalizable for player \(i\) for some type \(t_i \in T^\varepsilon_i(z_n)\); that is,

\[
\overline{p}^n(a_i) = P^n(\{z_n : a_i \text{ is weakly } \mathcal{C}(z_n)\text{-rationalizable}\}). \tag{3}
\]

The confidence set for prediction of \(a_i\) given \(n\) observations is \([p^n(a_i), \overline{p}^n(a_i)]\).

\(^{13}\)For example, consider a matching pennies game where player 1 receives \(\theta\) if players match and \(-\theta\) otherwise, and player 2 receives \(-\theta\) if the players match and \(\theta\) otherwise. Let \(\theta \in \{-1, 1\}\). Then if player 1 assigns probability 1 to \(\theta = 1\) while player 2 assigns probability 1 to \(\theta = -1\), it is (somewhat counterintuitively) a Bayesian Nash equilibrium for both players to choose match. See Dekel et al. (2004) for an extended discussion.
The larger $\overline{p}(a_i)$ and $\overline{p}(a_i)$ are, the more confident an analyst should be predicting that $a_i$ is rationalizable. At extremes: If $\overline{p}(a_i) = \overline{p}(a_i) = 1$, then given observation of $n$ random samples, the action $a_i$ is guaranteed to be rationalizable for player $i$ (for all permitted types). If $\overline{p}(a_i) = \overline{p}(a_i) = 0$, then action $a_i$ is guaranteed to not be rationalizable for player $i$ (for any permitted types). In the intermediate cases, if $0 < \overline{p}(a_i) = \overline{p}(a_i) < 1$, then rationalizability of the action $a_i$ depends on the specific realization of the data, and if $\overline{p}(a_i) < \overline{p}(a_i)$, then the prediction requires assumptions on the details of the agent’s belief beyond Assumption 1. I do not comment here on what further assumptions may be imposed, interpreting this case simply as one of ambiguity.

Some basic observations include:

**Observation 1.** For every player $i$ and action $a_i \in A_i$:

(a) $\overline{p}(a_i) \leq \overline{p}(a_i)$ for every $n \in \mathbb{Z}_+$.

(c) If $\mathcal{M}$ consists of a single learning rule, then $\overline{p}(a_i) = \overline{p}(a_i)$ for every $n \in \mathbb{Z}_+$.

Additionally, in the special case in which agents have a common prior, the definitions in $\overline{p}(a_i)$ and $\overline{p}(a_i)$ have the following familiar interpretation:

**Example 1.** (Common Prior.) Suppose that players share a common and correct prior over $\Theta \times \mathbb{Z}^\infty$. Write $\mu$ for the learning rule that maps $z_n$ into the induced posterior belief over $\Theta$ under the common prior. Then, each realization $z_n$ determines an interim game, where players all have common certainty in the posterior belief. Moreover, the common prior determines a distribution over data sets $z_n$, and hence a distribution over possible interim games. For any player $i$ and action $a_i$, the probabilities $\overline{p}(a_i) = \overline{p}(a_i)$, and are equal to the measure of size-$n$ datasets $z_n$ (under the common prior) with the property that action $a_i$ is rationalizable for player $i$ in the corresponding interim game.\(^{14}\)

In the above approach, the common prior serves multiple roles: it simultaneously determines the true distribution over the data that agents might see, and also determines how agents update from that data. When we separate these roles, we can still use an objective

\(^{14}\)This approach is similar for example to Kajii and Morris (1997) (if we re-interpret the histories $z_n$ as the states), where an incomplete information game is “close” to a complete information game if the payoffs of the complete information game occur with high probability under the prior.
data-generating process to define a measure over interim games, as I do here. In this way, the probabilities $p^n(a_i)$ and $p^a(a_i)$ are a natural generalization of a standard measure of the “typicality” of a strategic prediction, in the absence of a common prior.

4 Second Example: Coordination

To illustrate the definitions above, I now apply these ideas to a second classic setting: a two-player coordination game.

Suppose that a contagious disease spreads across a population at an unknown speed. Two states are connected, in that people travel between them, and their governors each choose between implementing a strong or a weak lockdown policy in their states to slow the spread of the disease. Implementation of the strict lockdown policy entails a large economic cost, but if the states coordinate on doing so, then the disease will be suppressed with certainty.

The two governors form beliefs about the growth rate of the disease based on a public data set $\{ (t, y_t) \}_{t=1}^n$, which consists of the number of reported cases of the disease, $y_t$, on days $t = 1, 2, \ldots, n$. The number of reported cases grows exponentially according to

$$\log y_t = \beta t + \varepsilon_t$$

where the noise term $\varepsilon_t$ has a normal distribution with known parameters $\mu = 0$ and $\sigma^2 > 0$. The constant $\beta$ is not known. Payoffs are given by the following matrix

<table>
<thead>
<tr>
<th></th>
<th>Strong</th>
<th>Weak</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong</td>
<td>$-1, -1$</td>
<td>$-1 - \beta, -\beta$</td>
</tr>
<tr>
<td>Weak</td>
<td>$-\beta, -1 - \beta$</td>
<td>$-\beta, -\beta$</td>
</tr>
</tbody>
</table>

The economic cost of the strong lockdown is normalized to 1, and the cost of letting the disease progress without a strong lockdown is given by its growth rate $\beta$. A weak lockdown is strictly dominant if $\beta < 1$, but coordination on the strong lockdown is the Pareto-dominant Nash equilibrium if $\beta > 1$.

Since the data is public, the assumption of a common prior over the model parameter $\beta$ necessarily restricts the agents to identical posterior beliefs. This rules out, for example, the possibility that a governor chooses the weak policy, not because he infers from the data that the disease is low-risk, but because he believes that the other governor makes such an inference.
In practice, the agents may use different statistical procedures for inferring $\beta$ from the data, leading to different estimates. For simplicity, I relax the assumption of a common prior in the following way. Define $\hat{\beta}(z_n)$ to be the ordinary least-squares estimate of $\beta$ from the data $z_n := \{(t, \log y_t)\}_{t=1}^n$, and let $\phi_n$ be the constant such that $C(z_n) = \{\beta : |\beta - \hat{\beta}(z_n)| \leq \phi_n\}$ is a 95%-confidence interval for $\beta$. Let $M$ be the set of all maps from the data $z_n$ into a distribution over the confidence interval $C(z_n)$. This allows players to disagree given the data, but requires the size of that disagreement to be confined within the confidence interval.

**Claim 2.** Suppose the true value of $\beta$ satisfies $\beta > 1$. Then, for every $\sigma^2 > 0$, both $p^n$ and $\bar{p}^n$ are increasing in $n$, while for every $n$, both $\underline{p}^n$ and $\bar{p}^n$ are decreasing in $\sigma^2$.

That is, suppose the actual growth rate is fast ($\beta > 1$), so that the strong lockdown is rationalizable given complete information of the payoffs. Then, the analyst gains confidence in this prediction as the reporting noise $\sigma^2$ decreases, and the number of observations $n$ increases.\footnote{It is straightforward to show that if instead $\beta < 1$, then the reverse statements hold; that is, the probabilities $\bar{p}^n$ and $\bar{p}^n$ are decreasing in $n$ and increasing in $\sigma^2$.} Lemma 1 in the appendix explicitly characterizes $P^n$ and $\bar{p}^n$. Using those expressions, I plot in Figure 4 the behavior of these confidence sets for different levels of reporting noise $\sigma$.

![Figure 4](image-url)

**Figure 4:** The shaded area depicts confidence sets $[\underline{p}^n, \bar{p}^n]$ for the rationalizability of the strong lockdown given $n$ common observations, and allowing the reporting noise $\sigma$ to vary. In all panels, $\beta = 2$.

As the number of observations $n$ grows, both $\underline{p}^n$ and $\bar{p}^n$ increase as well. If the number of observations is large relative to the reporting noise, then the analyst should have high
confidence that the strong lockdown policy is rationalizable—for example, if $\sigma = 10$ and $n = 100$, then the confidence set $[0.99, 1]$ is nearly degenerate at certainty. On the other hand, if the reporting noise is large and the number of observations is relatively small, then the strong lockdown is likely to be rationalizable for some permitted types, but not for all of them—for example, if $\sigma = n = 100$, then the confidence set is $[0.08, 0.99]$, suggesting substantial ambiguity regarding whether the strong lockdown is a good prediction of play.

5 Asymptotic Results

In some cases, such as in the examples in Sections 2 and 4, it is possible to explicitly characterize $[\overline{p}^n(a_i), \overline{p}^n(a_i)]$ for all $n$. I subsequently provide more general results for the behavior of these confidence sets, which are informative even when these characterizations are not available.

I first consider the limiting behavior of the probabilities $\overline{p}^n(a_i)$ and $\overline{p}^n(a_i)$ as the quantity of data $n$ gets large. Recall that by Assumption 2, the expected parameters under the different learning rules converge to a limiting value $\theta^\infty$. Thus, the $n = \infty$ limit corresponds to the complete information game indexed to $\theta^\infty$. For any action $a_i$, whether the probabilities $\overline{p}^n(a_i)$ and $\overline{p}^n(a_i)$ are continuous at $n = \infty$ tells us how sensitive rationalizability of $a_i$ is to an assumption that agents have coordinated their beliefs using infinite data. When these probabilities are discontinuous at $n = \infty$, then the infinite-data prediction is fragile, and the analyst would make different predictions for arbitrarily large finite quantities of data.

Formally, define $\overline{p}^\infty(a_i) = \overline{p}^\infty(a_i) = 1$ if the action $a_i$ is rationalizable in the game indexed to $\theta^\infty$, and define $\overline{p}^\infty(a_i) = \overline{p}^\infty(a_i) = 0$ if $a_i$ is not rationalizable.

**Definition 3.** Say that the confidence set for action $a_i$ is asymptotically continuous if

$$\lim_{n \to \infty} [\overline{p}^n(a_i), \overline{p}^n(a_i)] = [\overline{p}^\infty(a_i), \overline{p}^\infty(a_i)]$$

5.1 Fragile Predictions

Whether the confidence set for action $a_i$ is asymptotically continuous turns out to depend crucially on whether the expected parameters under the different learning rules converge uniformly to $\theta^\infty$. 
Assumption 3 (Uniform Convergence).

\[
\lim_{n \to \infty} \sup_{\mu \in \mathcal{M}} |\mathbb{E}_{\mu(Z_n)}(\theta) - \theta^\infty| = 0 \quad P\text{-a.s.}
\]

Assumption 2 already implies that for each learning rule \( \mu \in \mathcal{M} \), the expected parameters \( \mathbb{E}_{\mu(Z_n)}(\theta) \) converge to \( \theta^\infty \) as the quantity of data \( n \) grows large. Assumption 3 strengthens this by requiring additionally that the speed of convergence does not vary too much across the different learning rules in \( \mathcal{M} \). Specifically, the sequence of expectations \( \{\mathbb{E}_{\mu(Z_n)}\} \) must converge to \( \theta^\infty \) (as \( n \to \infty \)) uniformly across \( \mu \in \mathcal{M} \).

A sufficient condition for Assumption 3 to hold is that the set of learning rules \( \mathcal{M} \) is finite. But failures of Assumption 3 occur for classes of learning rules that we may consider plausible. In particular, Assumption 3 fails if the class \( \mathcal{M} \) is too rich, as in the following example:

Example 2 (Rich Sets of Priors and Likelihoods). An unknown parameter \( v \) takes value in \( \{0, 1\} \). Players commonly observe a sequence of realizations from the set \( Z = \{0, 1\} \). Learning rules \( \mu_{\pi,q} \in \mathcal{M} \) are indexed to parameters \( \pi \in (0,1) \) and \( q \in (1/2, 1) \), where the parameter \( \pi \) is the prior probability of value 1, and \( q \) identifies the following signal structure:

\[
\begin{align*}
z &= 0 & z &= 1 \\
v &= 0 & q &= 1 - q \\
v &= 1 & 1 - q &= 1
\end{align*}
\]

Each rule \( \mu_{\pi,q} \) takes the observed sequence of signal outcomes into the posterior belief over \( \{0, 1\} \), updating from the associated prior and signal structure. The true data-generating process belongs to this class; that is, there exists some \( q^* \in (1/2, 1) \) such that if \( v = 0 \), then the distribution over the signal set \( \{0, 1\} \) is \( (q^*, 1 - q^*) \), and if \( v = 1 \), then the distribution is \( (1 - q^*, q^*) \).

Note that in this example, all learning rules lead to the same belief (that is, there is asymptotic agreement in the sense of Acemoglu et al. (2015)). But because the rate of this convergence cannot be uniformly bounded across the different learning rules, it is possible for the confidence set to be discontinuous at \( n = \infty \).

Claim 3. Consider the trading game described in Section 2, and suppose that the data-generating process and set of learning rules are as given in Example 2. Then,

\[
\lim_{n \to \infty} [p^a_n(a_i), p^a_n(a_i)] = [0, 1]
\]
while \( [\mathbf{p}^c(a_i), \mathbf{p}^c(a_i)] = \{0\} \), so the prediction that entering is not rationalizable for the Seller is not asymptotically continuous.

The claim tells us that although trade will not occur in the limiting game, this prediction is sensitive to the assumption that agents have indeed coordinated their priors using infinite data. That is, even if the amount of data that players commonly observed were to be arbitrarily large, the analyst should nevertheless consider trade to be a plausible outcome (for the given set of learning rules).

### 5.2 Asymptotic Continuity

In contrast, when the assumption of uniform convergence is satisfied, then asymptotic confidence sets can be more tightly linked to predictions in the limiting complete information game.

**Theorem 1.** Suppose Assumption 3 is satisfied.

(a) If \( a_i \) is strictly rationalizable\(^{16} \) in the complete information game indexed to \( \theta^x \), then

\[
\lim_{n \to \infty} [p^n(a_i), \bar{p}^n(a_i)] = \{1\}.
\]

(b) If \( a_i \) is not rationalizable in the complete information game indexed to \( \theta^x \), then

\[
\lim_{n \to \infty} [p^n(a_i), \bar{p}^n(a_i)] = \{0\}.
\]

This proposition says that if an action \( a_i \) is strictly rationalizable in the limiting complete information game (given infinite samples), then \( \bar{p}^n(a_i) \) and \( \underline{p}^n(a_i) \) both converge to 1 as \( n \) grows large. Thus, when agents observe sufficiently large quantities of public data, the analyst should be arbitrarily confident in predicting that \( a_i \) is rationalizable. On the other hand, if action \( a_i \) is not rationalizable in the limiting game, then \( \bar{p}^n(a_i) \) and \( \underline{p}^n(a_i) \) both converge to 0, so the analyst should be arbitrarily confident in predicting that \( a_i \) is not rationalizable for large data sets.

\(^{16}\)Recall that an action is *strictly rationalizable* in a complete information game if there exists a family of sets \( (R_j)_{j \in J} \subseteq \prod_{j \in J} A_j \) such that \( a_i \in R_i \), and for each player \( j \) and action \( a_j \in R_j \), there is a belief \( \alpha_{-j} \in \Delta(R_{-j}) \) to which \( a_j \) is a strict best reply.
The intermediate case in which $a_i$ is rationalizable at $\theta^\infty$, but not strictly rationalizable, is subtle and depends on details of the game. An example is given in Appendix D.2 in which $\lim_{n \to \infty} [p^n(a_i), \tilde{p}^n(a_i)] = \{1\}$, and a second example is given in Appendix D.3.2 in which $\lim_{n \to \infty} [p^n(a_i), \tilde{p}^n(a_i)] = [0, 1]$. Note that the latter corresponds to a maximally ambiguous outcome—no amount of data is decisive on whether or not the action should be considered rationalizable. These subtleties are discussed at length in Appendix D, where I provide a necessary condition for the confidence set to converge to certainty, namely that the action $a_i$ is "weakly" strictly-rationalizable at $\theta^\infty$, a property that I define.

I now provide a brief explanation for what determines these different asymptotic behaviors. A key definition is the set of complete information games in which $a_i$ is rationalizable:

**Definition 4.** Let $\Theta^{a_i} \subseteq \Theta$ be the set of parameters $\theta$ such that $a_i$ is rationalizable for player $i$ in the complete information game indexed to $\theta$.

Under Assumption 3, as the quantity of data $n$ gets large, players eventually have common certainty of a shrinking neighborhood around the limiting parameter $\theta^\infty$. If $\theta^\infty \notin \Theta^{a_i}$, so that action $a_i$ is not rationalizable in the limiting game, then players eventually have common certainty of a set that is disjoint from $\Theta^{a_i}$. The action $a_i$ clearly cannot be rationalizable for all types satisfying this common certainty restriction, so $p^n(a_i) \to 0$. Part (b) of Theorem 1 strengthens this by showing that $a_i$ is eventually not rationalizable for any type satisfying this restriction, so also $\tilde{p}^n(a_i) \to 0$.

Now suppose that the limiting parameter $\theta^\infty$ is in the interior of the set $\Theta^{a_i}$, implying that players eventually have common certainty of a shrinking set in $\Theta^{a_i}$. A natural conjecture is that the confidence set $[p^n(a_i), \tilde{p}^n(a_i)]$ must then converge to 1. But common certainty in subsets of $\Theta^{a_i}$—indeed, common certainty in arbitrarily small open sets within $\Theta^{a_i}$—does not guarantee rationalizability of $a_i$, as the example in Section D.3.2 demonstrates.

I show that if, however, players have common certainty of a set $\mathcal{C}$ on which action $a_i$ is rationalizable using the same chain of best replies, then $a_i$ must be rationalizable. Formally, say that the family of sets $(R_j)_{j \in \mathcal{I}} \subseteq \prod_{j \in \mathcal{I}} A_j$ and beliefs $(\nu[a_j])_{a_j \in R_j, j \in \mathcal{I}}$ rationalize the action $a_i$ at $\theta$ if for every player $j$, each $a_j \in R_j$ is a best reply to the belief $\nu[a_j]$ in the game indexed to the parameter $\theta$. Further define:

---

17 The dependence of $\Theta^{a_i}$ on the player index $i$ is dropped to save notation.
18 A nice example in the concurrent work of Chen and Takahashi (2017) shows this as well.
19 Recall that in any complete information game $\theta$, the family of sets $(R_j)_{j \in \mathcal{I}} \subseteq \prod_{j \in \mathcal{I}} A_j$ is closed under
Definition 5. Say that the action $a_i$ can be rationalized using the same chain of best replies on $C \subseteq \Theta$ if there exists a tuple $(R_j, (\nu[a_j])_{a_j \in R_j})_{j \in I}$ that rationalizes $a_i$ at every $\theta \in C$.

I show in Lemma 5 that if $a_i$ can be rationalized using the same chain of best responses on a set $C$, then it must be rationalizable for all types with common certainty of $C$.

Since strict rationalizability of the action $a_i$ at the limiting parameter $\theta^\infty$ implies that it can be rationalized using the same chain of best replies on a neighborhood of $\theta^\infty$, Part (a) of Theorem 1 follows.

I conclude with a remark on how these results connect to the literature regarding topologies over the universal type space. Under Assumption 3, it can be shown that the set of types $T_i^{e(\mathbb{E}_n)}$ $P$-almost surely converges to the singleton type with common certainty in the limiting parameters $\theta^\infty$, where this convergence is in the Hausdorff metric induced by the uniform-weak metric (Chen et al., 2010) on the universal type space. It is crucial that convergence occurs in the uniform-weak metric; indeed, this is what allows refinement to be obtained despite the negative results of Weinstein and Yildiz (2007). It is also crucial that convergence is “uniform” across the set, as implied by convergence in the Hausdorff metric. Indeed, we know from Chen et al. (2010) that strict rationalizability is lower hemi-continuous in the uniform-weak topology. Thus, if the action $a_i$ is strictly rationalizable at the infinite-data limit, then (with probability 1 over sample paths) it must eventually be rationalizable along any sequence of types $t_i^n$ from $T_i^{e(\mathbb{E}_n)}$. But this does not imply strong $C$-rationalizability, which requires that $a_i$ is rationalizable for all types from $T_i^{e(\mathbb{E}_n)}$ when $n$ is sufficiently large.

---

20 An immediate implication is that strict rationalizability is stronger than the condition that $\theta^\infty$ is in the interior of $\Theta^{a_i}$. I study this further in Appendix D.4, and provide a condition which characterizes the interior of this set, namely that the action $a_i$ is “weakly-strict rationalizable.”

21 Part (a) is closely related to Morris et al. (2012), as the property that $\mathbb{P}^\nu(a_i) \rightarrow 1$ is very similar to the property that $a_i$ is robustly rationalizable, as defined in Morris et al. (2012). A key difference is that Morris et al. (2012) consider almost common belief in the exact parameter $\theta^\infty$, while I consider common certainty in a neighborhood of $\theta^\infty$. Nevertheless, as Proposition 1 in Morris et al. (2012) shows, strict rationalizability is also a sufficient condition for robust rationalizability.

22 The main result in Weinstein and Yildiz (2007) relies on types that converge only in the (coarser) product topology. Informally, the assumption that players have common certainty in a set of first-order beliefs imposes crucial discipline on tail beliefs, ruling out cases such as constructed in Weinstein and Yildiz (2007).
The stronger property that types converge *uniformly* over the set $T_i^{\infty(z_n)}$ delivers the desired result, and the discussion above shows how this uniform convergence established.

6 (Small) Finite Samples

The previous section described confidence sets given large numbers of common observations. I now focus on the setting of small $n$, and bound the extent to which the agent’s confidence set, $[\underline{P}^n(a_i), \overline{P}^n(a_i)]$, diverges from its asymptotic limit $[\underline{P}^\infty(a_i), \overline{P}^\infty(a_i)]$. Throughout this section, I impose the simplifying assumptions that observations are i.i.d., and that they take values from a finite set $Z$:

**Assumption 4.** $Z_1, \ldots, Z_n \sim \text{i.i.d. } Q$.

**Assumption 5.** $|Z| < \infty$.

In some cases, as in the examples given in Section 2, the confidence sets can be exactly characterized. In Section 6.1, I provide bounds for the confidence set that can be easier to derive in certain cases, and the subsequent Section 6.2 provides comparative statics for how the confidence set varies with properties of the set of learning rules.

6.1 Quantitative Bounds for Confidence Sets

First consider an action $a_i$ that is strictly rationalizable in the limiting game $\theta^\infty$. Theorem 1 tells us that the analyst’s confidence set $[\underline{P}^n(a_i), \overline{P}^n(a_i)]$ converges to a degenerate interval at 1. Theorem 2, below, provides a lower bound on $\underline{P}^n(a_i)$, which informs how fast this convergence occurs.

A key input into the bound is the “degree” to which $a_i$ is strictly rationalizable in the limiting game. Say that an action $a_i$ is $\delta$-strictly rationalizable for player $i$ in game $\theta$ if there exists a family of sets $(R_j)_{j \in \mathcal{Z}}$ with $a_i \in R_i$, such that for every player $j$ and action $a_j \in R_j$, there is some distribution $\alpha_{-j} \in \Delta(R_{-j})$ where

$$u_j(a_j, \alpha_{-j}, \theta) - \delta > u_j(a'_j, \alpha_{-j}, \theta) \quad \forall a'_j \neq a_j.$$  \hspace{1cm} (4)

That is, each $a_j$ is a strict best-reply to $\alpha_{-j}$ with a slack of at least $\delta$.\footnote{This definition mirrors the more classic notion of $\epsilon$-best reply.} Now define

$$\delta^\infty := \sup \{ \delta : a_i \text{ is } \delta\text{-strictly rationalizable in the game } \theta^\infty \}.$$
Then, if $a_i$ is strictly rationalizable in the limiting complete information game, and players have commonly observed $n$ realizations, the probability that $a_i$ is rationalizable for all permitted types can be upper bounded as follows.

**Theorem 2.** Suppose $a_i$ is strictly rationalizable for player $i$ in the game $\theta^\infty$. Then, for every $n \geq 1$,

$$p^n(a_i) \geq 1 - \frac{K}{\delta^\infty} \mathbb{E} \left[ \sup_{\theta' \in \mathcal{E}(Z_n)} \|\theta' - \theta^\infty\|_\infty \right]$$

(5)

where $K$ is the Lipschitz constant of the map $g : \Theta \to U$.

Recalling that $p^n(a_i) \geq p^n(a_i)$ for every $n$, this proposition allows us to lower bound the confidence set $[p^n(a_i), \bar{p}^n(a_i)]$.

The expression in (5) is increasing in $\delta^\infty$, so the “more strictly-rationalizable” the action is in the limiting game, the fewer observations are necessary for the prediction to hold. The bound is decreasing in $\mathbb{E} \left[ \sup_{\theta' \in \mathcal{E}(Z_n)} \|\theta' - \theta^\infty\|_\infty \right]$, which is the expected distance from the limiting parameter $\theta^\infty$ to the farthest parameter in the plausible set $\mathcal{E}(Z_n)$. When Assumption 3 is satisfied, then $\mathbb{E} \left[ \sup_{\theta' \in \mathcal{E}(Z_n)} \|\theta' - \theta^\infty\|_\infty \right] \to 0$ as $n \to \infty$, and the speed of this convergence can be interpreted as the speed at which players commonly learn (Cripps et al., 2008). Thus, Theorem 2 tells us that the quicker players commonly learn, the fewer observations are necessary for limiting predictions to carry over to small-data settings.

This qualitative finding is not so surprising, but in some cases the expression in (5) can be quantified, as in the following example:

**Example 3.** Consider the payoff matrix from Section 4 with unknown parameter $\beta \in \mathbb{R}$. Suppose that players commonly observe $n$ public signals $z_t = \beta + \varepsilon_t$, with standard normal error terms $\varepsilon_t$ that are i.i.d. across observations. The set of possible learning rules in $\mathcal{M}$ are identified with different prior beliefs $\beta \sim \mathcal{N}(x, 1)$, where $x$ is in the bounded interval $[-\eta, \eta]$. Let the true value of $\beta$ satisfy $\beta > 1$. Then:

**Corollary 1.** For each $n \geq 1$,

$$p^n(\text{strong}) \geq 1 - \frac{1}{\beta - 1} \left( \sqrt{\frac{2}{\pi n}} + \frac{\beta + \eta}{n + 1} \right)$$

The bound in Corollary 1 is decreasing in $\eta$ (the size of the model class), increasing in $n$ (the number of observations), and increasing in $\beta - 1$ (the strictness of the solution at the limit).
Now suppose that the action $a_i$ is not rationalizable in the limiting game $\theta^\infty$. We know from Part (c) of Theorem 1 that in this case, the analyst’s confidence set $[\bar{p}^n(a_i), \overline{p^n}(a_i)]$ converges to a degenerate interval at zero. But given small quantities of data $n$, the action $a_i$ may still constitute a plausible prediction of play, as in the trading game studied in Section 2. Claim 4, below, provides an upper bound on $\overline{p^n}(a_i)$, which informs whether the analyst should consider $a_i$ a plausible prediction away from the limit.

To define this bound, a few intermediate definitions are needed. Let $Z_{a_i}$ be all data sets $z_n$ given which the action $a_i$ is weakly $\mathcal{C}(z_n)$-rationalizable. (This set is characterized in Lemma 6 in the appendix, but must be determined on a case-by-case basis.) Each data set $z_n$ is associated with an empirical measure $\hat{Q}_{z_n}$ on $\Delta(Z)$. The Kullback-Leibler distance between any such distribution $\hat{Q}_{z_n}$, and the actual data-generating distribution $Q$, is $D_{KL}(\hat{Q}_{z_n} || Q) = \sum_{z \in Z} Q(z) \log \left( \frac{Q(z)}{\hat{Q}_{z_n}(z)} \right)$. Finally, define

$$Q^*_n = \arg\min_{\hat{Q} \in \{z_n \in Z_{a_i} \}} D_{KL}(\hat{Q}_{z_n} \| Q)$$

to be the empirical measure associated with a data set in $Z_{a_i}$ that is closest in Kullback-Leibler distance to $Q$. Application of Sanov’s theorem directly gives the following corollary.

Claim 4. Suppose $a_i$ is not rationalizable for player $i$ in game $\theta^\infty$; then, for every $n \geq 1$,

$$\overline{p}_n \leq (n + 1)^{|Z|} 2^{-nD_{KL}(Q^*_n || Q)}.$$

This bound is illustrated below in an example setting:

Example 4. Consider the trading game from Section 2 and the learning rules described in Example 2, but change the domain of $q$ to $[2/3, 1]$ and the domain of $\pi$ to $[1/4, 3/4]$. Suppose that the true signal structure is identified with $q^* = 3/4$. We know from Theorem 1 that entering will fail to be rationalizable when players have observed sufficient data. Nevertheless, the action may be rationalizable for a permitted belief if players have observed a small number of data points, and the corollary below quantifies this.

Corollary 2. For each $n \geq 1$,

$$\overline{p}^n(\text{enter}) \leq (n + 1)^{|Z|} 2^{-rn}$$

where $r = \frac{3}{4} \left( \log(3n) - \log \left( \frac{n}{2} + \frac{\log(9)}{\log(2)} \right) \right) + \frac{1}{4} \left( \log(n) - \log \left( \frac{n}{2} + \frac{\log(9)}{\log(2)} \right) \right).$
6.2 Comparative Statics in the Set of Learning Rules

In this final section, I provide qualitative comparisons across different sets of learning rules. To facilitate these comparisons, it is useful to write \( [p^n_M(a_i), \overline{p}^n_M(a_i)] \), making explicit the dependence of the confidence set on \( M \).

A first observation is that expanding the set of learning rules leads to larger, or more “ambiguous,” confidence sets:

**Observation 2.** Suppose \( M \subseteq M' \). Then, \( [p^n_M(a_i), \overline{p}^n_M(a_i)] \subseteq [p^n_{M'}(a_i), \overline{p}^n_{M'}(a_i)] \).

This is because the set of expected parameters \( C(z_n) \) weakly expands as the set of learning rules grows larger, and so the set of permitted interim types \( T^C(z_n) \) expands also. The condition that \( a_i \) is rationalizable for all types in \( T^C(z_n) \) becomes more difficult to fulfill, reducing the lower bound \( p^n(a_i) \), and the condition that \( a_i \) is rationalizable for some type in \( T^C(z_n) \) becomes easier to fulfill, increasing the upper bound \( \overline{p}^n(a_i) \).

More generally, the greater the diversity in expected parameters induced by learning rules in \( M \), the smaller the probability \( p^n(a_i) \) becomes. Proposition 1 formalizes this by fixing the distribution over beliefs induced by each individual learning rule in \( M \), and allowing for arbitrary joint distributions \( (\mu(Z_n))_{\mu \in M} \) subject to this constraint.

The simple example below previews the bounds in the subsequent proposition. Consider the following coordination game:

\[
\begin{array}{ccc}
a_3 & a_4 \\
a_1 & \theta, \theta & \theta - 1, 0 \\
a_2 & 0, \theta - 1 & 0, 0
\end{array}
\]

where \( \theta \in \{1, -1\} \), and suppose that players learn about \( \theta \) based on a random variable \( Z \in \mathbb{R} \). Allow for any set of learning rules \( M = \{\mu_1, \mu_2\} \), where the distributions of the individual learning rules are fixed to be

\[
\begin{align*}
\mu_1(Z) & \sim \frac{3}{4}(q, 1 - q) + \frac{1}{4}(\overline{q}, 1 - \overline{q}) \\
\mu_2(Z) & \sim \frac{2}{3}(\overline{q}, 1 - \overline{q}) + \frac{1}{3}(q, 1 - q)
\end{align*}
\]

for some \( q, \overline{q} \in [0, 1] \) satisfying \( \overline{q} > 1/2 > q \). That is, the realization of \( Z \) is mapped into either of two beliefs: a more optimistic belief \((\overline{q}, 1 - \overline{q})\) and a more pessimistic belief \((q, 1 - q)\). The ex-ante probability of the optimistic belief is 3/4 under learning rule \( \mu_1 \), and 2/3 under \( \mu_2 \). What is the range of possible values for \( p^n_{M'}(a_1) \)?
In this game, the action $a_1$ is strongly $\mathcal{C}(z)$-rationalizable for player 1 if and only if $\mu_1(z) = \mu_2(z) = (\bar{q}, 1 - \bar{q})$. Thus, it can be shown that $p_n(a_1)$ is largest when the learning rules maximally overlap in the realizations of $Z$ that are sent to the optimistic belief, in which case the probability $p_n(a_1)$ is the lesser probability of the optimistic belief, namely $2/3$. At the other extreme, the probability $p_n(a_1)$ is minimized when $\mu_1(Z)$ and $\mu_2(Z)$ are “anti-correlated,” so that the realizations of $Z$ taken to the optimistic belief overlap as little as possible. In this case, $p_n = 1 - 1/4 - 1/3 = 5/12$.

These observations can be generalized as follows for arbitrary finite $\mathcal{M}$ under the following technical assumption.

**Assumption 6.** Action $a_i$ can be rationalized under the same chain of best replies on $\Theta^{a_i}$, the set defined in Definition 4.

The assumption is needed for reasons related to the discussion in Section 6, and is satisfied in the above coordination game.

**Proposition 1.** Suppose Assumption 6 is satisfied. For each $n$, fix arbitrary distributions $Q^n_1, \ldots, Q^n_K \in \Delta(\Delta(\Theta))$ and suppose that $\mathcal{M} = \{\mu_1, \ldots, \mu_K\}$ has the property that each $\mu_k(Z_n) \sim Q^n_k$. Then,

$$1 - K + \sum_{\mu \in \mathcal{M}} p^n_\mu(a_i) \leq p^n_\mathcal{M}(a_i) \leq \min_{\mu \in \mathcal{M}} p^n_\mu(a_i)$$

where each $p^n_\mu(a_i)$ is the probability associated with the set $\mathcal{M} = \{\mu\}$.

The upper bound corresponds to the case in which different learning rules overlap as much as possible in the data sets that are mapped to $\Theta^{a_i}$. That is, if $a_i$ is rationalizable in the complete information game $\mathbb{E}_{\mu(z_n)}(\theta)$ for some $\mu \in \mathcal{M}$, then it is likely to be rationalizable also in every other complete information game $\mathbb{E}_{\mu'(z_n)}(\theta)$, $\mu' \in \mathcal{M}$. The lower bound, when attainable, corresponds to the case in which different learning rules disagree as much as possible in which data sets are mapped to $\Theta^{a_i}$. This intuition can be formalized using the idea of co-monotonic and counter-monotonic random variables, and the proof follows from a straightforward application of the Frechet-Hoeffding bound.

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24Suppose $\mu(z) = (\bar{q}, 1 - \bar{q})$ for some $\mu \in \mathcal{M}$. Then, $\mathbb{E}_\mu(\theta) < 1/2$ and the action $a_1$ is strictly dominated for the player with common certainty that $\theta$ has this value.

25The set $\Theta^{a_1} = \mathbb{R}_+$, and $a_1$ is rationalizable on it using the tuple $\{a_1, \{a_3\}, \delta_{a_3}, \delta_{a_1}\}$, where $\delta_{a_3}$ and $\delta_{a_1}$ are, respectively, degenerate distributions on $a_3$ and $a_1$. 

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7 Extensions

In this section, I briefly discuss how certain assumptions that are made in the main text can be relaxed.

Limit Misspecification and Disagreement. In the main text, I imposed an assumption which guaranteed that learning rules uniformly recovered the true parameter $\theta^x$ as the quantity of data grew large. It is possible to replace Assumption 3 with the following, weaker property, which allows players to have heterogeneous and incorrect beliefs even in the limit: For any $\epsilon > 0$, say that the class of learning rules $\mathcal{M}$ satisfies $\epsilon$-Uniform Convergence if

$$\lim_{n \to \infty} \sup_{\mu \in \mathcal{M}} |\mathbb{E}_{\mu(Z_n)}(\theta) - \theta^x| \leq \epsilon \quad P\text{-a.s.}$$

This requires that the set of expected parameters converges to an $\epsilon$-neighborhood of $\theta^x$. Then, Theorem 1 holds so long as the set of learning rules $\mathcal{M}$ satisfies $\epsilon$-Uniform Convergence for $\epsilon$ sufficiently small. For example, Part (a) of Theorem 1 holds so long as $\mathcal{M}$ satisfies $\delta^x/K$-Uniform Convergence, where $K$ is the Lipschitz constant of the map $g : \Theta \to U$. The rate results do not change.

Approximate Common Certainty. Suppose that instead of imposing common certainty in $\mathcal{C}(z_n)$, as we have done in the main text, we assume that players have common $p$-belief in $\mathcal{C}(z_n)$. In this case, the rate results in Section 6 require modification, but all asymptotic results stated in Section 5 extend for $p$ sufficiently large. Specifically, suppose that the type set $T_i^{p}(z_n)$ is replaced by the set of all types that have common $p$-belief in the set $\mathcal{C}(z_n)$. Then, there exists a $\bar{p}$ such that so long as players have common $p$-belief in $\mathcal{C}(z_n)$, where $p > \bar{p}$, then Theorem 1 holds as stated. This extension relies on the set of payoff functions $g(\Theta)$ being bounded, as was assumed in Section 3.1. Rate results similar to those in Section 6 can also be obtained, but will be different from the ones reported here.

Partially Identified $\theta$. I have assumed so far that the limit as $n$ grows large is a complete information game. The proposed approach can be extended in a simple way so that the limit game is instead a game of incomplete information. Let $\nu \in \Delta(\Theta)$ be the “true” distribution over uncertainty (a “limit common prior”) and rewrite Assumption 3 as follows:

$$\sup_{\mu \in \mathcal{M}} d(\mu(Z_n), \nu) \to 0 \quad P\text{-almost surely,}$$

where $d$ is the Prokhorov metric on $\Delta(\Theta)$. Then,
defining $\theta^x := \mathbb{E}_\nu(\theta)$ to be the expected parameter under $\nu$, all the results in Section 5 follow without revision. Note that the interpretation of $\theta^x$ is different in this case—that is, it no longer represents the “true” complete information game, but rather the game that is perceived given the partial data that is available.

Confidence Sets for Equilibrium. The proposed approach can be paired with solution concepts besides rationalizability. For example, suppose we are interested in evaluating an analyst’s confidence in predicting that the action profile $a = (a_1, \ldots, a_I)$ is part of a Bayesian Nash equilibrium, when agents have commonly observed $n$ datapoints. The analogous confidence set is $[\overline{p}^n(a), \underline{p}^n(a)]$, where the lower bound $\overline{p}^n(a)$ is the probability (over possible datasets $z_n$) that $a_i$ is a best reply to $a_{-i}$ for every player $i$ of any type $t_i \in T_i(z_n)$. The upper bound $\underline{p}^n$ is the probability that there exists some belief-closed type space $(T_i, \kappa_i)_{i \in I}$ where each $T_i \subseteq T_i^{\kappa_i(z_n)}$, and the strategy profile $\sigma$ with $\sigma_i(t_i) = a_i$ for all $i, t_i \in T_i$ is a Bayesian Nash equilibrium. All of the main results have analogues for this case (for example, Theorem 1 holds with “strict rationalizability” replaced with “strict equilibrium” in the limiting game).

8 Conclusion

Economists make predictions in incomplete information games based on specific beliefs assigned to agents, but we don’t know if those are the beliefs actually held by those agents. A large literature on the robustness of strategic predictions to the specification of agent beliefs provides guidance regarding whether these predictions should be trusted. These classic robustness notions tend to be qualitative, i.e. we learn whether the prediction is or isn’t robust to perturbations in the agents’ beliefs. Here I offer a different perspective, namely a quantitative metric for how robust the prediction is. The metric depends on the quantity of data that agents get to see. Predictions that hold given infinite quantities of data may not hold given large quantities of data, and those that hold given large quantities of data may not hold in environments where agents see only a few observations. Likewise, predictions that don’t hold at the limit may nevertheless be plausible when agents’ beliefs are coordinated by a small number of observations. The proposed framework provides a way of formalizing this, generating new comparative statics for how the analyst’s confidence in a strategic prediction
varies with primitives of the learning environment.

References


Appendix

A Proofs for Section 2

A.1 Proof of Claim 2

I first demonstrate the following lemma, which explicitly characterizes the probabilities $p^n$ and $\overline{p}^n$.

**Lemma 1.** For every $n \geq 1$,

$$p^n = 1 - \Phi \left( -1.96 - \frac{\beta - 1}{\sigma} \sqrt{\frac{n^2 - 1}{12}} \right)$$

while

$$\overline{p}^n = 1 - \Phi \left( 1.96 - \frac{\beta - 1}{\sigma} \sqrt{\frac{n^2 - 1}{12}} \right)$$

Since $\beta > 1$ by assumption, both expressions are decreasing in $\sigma$ and increasing in $n$. Thus Claim 2 directly follows. Towards this result, I first prove the following intermediate lemma:

**Lemma 2.** The strong policy is rationalizable

(a) for all types with common certainty in $\{\beta \in \mathcal{C}\}$ if and only if the set $\mathcal{C} \subseteq [1, \infty)$

(b) for some type with common certainty in $\{\beta \in \mathcal{C}\}$ if and only if $\mathcal{C} \cap [1, \infty) \neq \emptyset$

**Proof.** For Part (a), $\mathcal{C} \subseteq [1, \infty)$ is a necessary condition, as otherwise there exists some $\beta' \in \mathcal{C} \backslash [1, \infty)$, and the strong policy is not be rationalizable for the type with common certainty in $\beta'$. That $\mathcal{C} \subseteq [1, \infty)$ is sufficient follows from Part (a) of the subsequent Lemma 5. For Part (b), suppose $\mathcal{C} \cap [1, \infty) = \emptyset$. In this case, the strong policy is strictly dominated for every type with common certainty in $\mathcal{C}$, since the expected payoff to the strong policy is strictly below $-1$. On the other hand, suppose $\mathcal{C} \cap [1, \infty) \neq \emptyset$, and choose any $\beta$ in this nonempty intersection. The strong policy is rationalizable for the type with common certainty in this value of $\beta$. □

I now prove Lemma 1.

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Proof. Lemma 2 implies that \( p^n = \Pr(\mathcal{C}(z_n) \subseteq [1, \infty)) \) while \( \overline{p}^n = \Pr(\mathcal{C}(z_n) \cap [1, \infty) \neq \emptyset) \), so it remains to determine the probability of these events. The (log-linearized) public data is \( \{(t, z_t)\}_{t=1}^n \), where \( z_t := \log y_t = \beta t + \log \varepsilon_t \), and \( \log \varepsilon_t \sim \mathcal{N}(0, \sigma^2) \). Using standard results for ordinary least-squares (Hastie et al., 2009), the distribution of the OLS estimator \( \hat{\beta} \) is

\[
\hat{\beta} \sim \mathcal{N}\left(\beta, \frac{\sigma^2}{\frac{1}{n} \sum_{t=1}^n (t - \bar{t})^2}\right)
\]

where \( \bar{t} = \frac{1}{n} \sum_{t=1}^n t \). Since

\[
\frac{1}{n} \sum_{t=1}^n (t - \bar{t})^2 = \frac{1}{n} \left( \sum_{t=1}^n t^2 - 2\bar{t} \sum_{t=1}^n t + \sum_{t=1}^n \bar{t}^2 \right) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{2} + \left( \frac{n+1}{2} \right)^2 = \frac{(n^2-1)}{12}
\]

we can simplify the variance of \( \hat{\beta} \) to \( \frac{12\sigma^2}{n^2-1} \). The 95% confidence interval for \( \beta \) is thus

\[
\left[ \hat{\beta} - 1.96\sigma \cdot \sqrt{\frac{12}{n^2-1}}, \hat{\beta} + 1.96\sigma \cdot \sqrt{\frac{12}{n^2-1}} \right]
\]

(6)

The probability that the interval in (6) is contained in \([1, \infty)\) is

\[
\Pr \left( \hat{\beta} > 1 + 1.96\sigma \cdot \sqrt{\frac{12}{n^2-1}} \right)
\]

which is equal to

\[
1 - \Phi \left( 1.96 - \frac{\beta - 1}{\sigma} \sqrt{\frac{n^2-1}{12}} \right)
\]

where \( \Phi \) is the standard normal cdf. By Part (a) of Lemma 2, this is equal to \( p^n \).

Likewise, the probability that the interval in (6) has nonempty intersection with \([1, \infty)\) is given by

\[
\Pr \left( \hat{\beta} > 1 - 1.96\sigma \cdot \sqrt{\frac{12}{n^2-1}} \right)
\]

which is equal to

\[
1 - \Phi \left( -1.96 - \frac{\beta - 1}{\sigma} \sqrt{\frac{n^2-1}{12}} \right)
\]

By Part (b) of Lemma 2, this is equal to \( \overline{p}^n \). \( \square \)
A.2 Proof of Claim 1

Since there is no noise in the data-generating process, the true classification rule \( f \) is always consistent with the realized observations. Let \( \pi \) be a prior degenerate on \( f \); given any realization of the data, the posterior will also be degenerate on \( f \), with expectation \( \mathbb{E}_\pi(v \mid z_n) = f(x_0) = 1 \). Hence, \( 1 \in \mathcal{C}(z_n) \) for every \( z_n \), so common certainty in \( v = 1 \) is always consistent with Assumption 1. But entering is not rationalizable for the Seller with this belief, implying \( p^n = 0 \) for every \( n \).

To prove Parts (a) and (b) of the claim, which refer to the probability \( p^n \), I first show that entering is rationalizable for some type satisfying Assumption 1 if and only if there exist rectangular classification rules that are consistent with the data, and which make conflicting predictions for the Seller’s good \( x_0 \) (Lemma 3). I characterize the probability of this event in Lemma 3, and the comparative statics for \( p^n \) follow immediately.

**Lemma 3.** Fix an arbitrary data set \( z_n = \{(x_i,f(x_i))\}_{i=1}^n \). Entering is rationalizable for the Seller with a belief satisfying Assumption 1 if and only if there exist \( \tilde{f}, \tilde{f}' \in \mathcal{F} \) where

1. \( \tilde{f}(x_i) = \tilde{f}'(x_i) = f(x_i) \) for each observation \( i = 1, \ldots, n \)
2. \( \tilde{f}(x_0) = 1 \) while \( \tilde{f}'(x_0) = 0 \)

Item (1) says that \( \tilde{f} \) and \( \tilde{f}' \) agree with the true \( f \) at the observed \( x_i \), and hence are consistent with the data. Item (2) says that \( \tilde{f} \) predicts \( v = 1 \) while \( \tilde{f}' \) predicts \( v = 0 \).

**Proof.** Suppose there exists a pair \( \tilde{f}, \tilde{f}' \) satisfying (1) and (2), and define priors \( \pi_{\tilde{f}}, \pi_{\tilde{f}'} \in \Delta(\mathcal{F}) \) that are degenerate at these classification rules. Since these rules are consistent with the data, the posterior beliefs updated to \( z_n \) are likewise degenerate at \( \tilde{f} \) and \( \tilde{f}' \), so the posterior expectations of \( v = f(x_0) \) are \( \mathbb{E}_{\pi_{\tilde{f}}}(v \mid z_n) = 1 \) and \( \mathbb{E}_{\pi_{\tilde{f}'}}(v \mid z_n) = 0 \), implying that \( \{0,1\} \subseteq \mathcal{C}(z_n) \). Entering is rationalizable for the Seller who believes that \( v = 1 \) with probability 1, and who believes with probability 1 that the Buyer believes that \( v = 0 \) with probability 1. This belief is consistent with common certainty in \( \mathcal{C}(z_n) \).

Now suppose that no such pair \( \tilde{f}, \tilde{f}' \) exists, implying either that every \( \tilde{f} \in \mathcal{F} \) consistent with the data predicts \( f(x_0) = 0 \), or that every \( \tilde{f} \in \mathcal{F} \) consistent with the data predicts \( f(x_0) = 1 \). Then either \( \mathcal{C}(z_n) = \{1\} \) or \( \mathcal{C}(z_n) = \{0\} \). If the former, the only type satisfying Assumption 1 is the one with common certainty in \( v = 1 \), and if the latter, the only type satisfying Assumption 1 is the one with common certainty that \( v = 1 \). Entering is not rationalizable for the Seller with either of these beliefs. \[\square\]
**Lemma 4.** Suppose the true function is \( f(x) = \mathbb{1}(x \in R) \) where \( R = [-\tau_1, \tau_1] \times [-\tau_2, \tau_2] \times \ldots [-\tau_m, \tau_m] \) for a sequence of constants \( \tau_1, \tau_1, \ldots, \tau_m, \tau_m \in (0, 1) \). Then

\[
\overline{p}^n(a_i) = 1 - \prod_{k=1}^{m} \left( 1 - \left( \frac{1}{2} \right)^n \left[ (2 - \tau_k)^n + (2 - \bar{\tau}_k)^n - (2 - (\tau_k + \bar{\tau}_k))^n \right] \right).
\]

**Proof.** From Lemma 3, the probability \( \overline{p}^n \) is equal to the measure of data sets \( z_n \) given which there exist rectangular classification rules \( \tilde{f}, \tilde{f}' \) that are consistent with \( z_n \), and which make conflicting predictions at the input \( x_0 \). The true classification rule \( f \) is always consistent with the data, and predicts \( f(x_0) = 0 \), so a pair of such rules exists if we can additionally find a rule \( \tilde{f} \in \mathcal{F} \) consistent with the data that predicts \( \tilde{f}(x_0) = 1 \).

A necessary and sufficient condition for existence of such a rule is that there is some dimension \( k \) on which either every observation \( x_i \) satisfies \( x_i^k < 0 \), or every \( x_i \) satisfies \( 0 < x_i^k \). This allows some \( \tilde{f} \in \mathcal{F} \) to be consistent with the data, but to predict \( 1 \) at the zero vector.

For each dimension \( k \), the probability that there is at least one observation \( x_i \) with \( x_i^k \in [-\tau_k, 0) \) and at least one observation \( x_j \) with \( x_j^k \in (0, \tau_k] \) is

\[
1 - \left( \frac{1}{2} \right)^n \left[ (2 - \tau_k)^n + (2 - \bar{\tau}_k)^n - (2 - (\tau_k + \bar{\tau}_k))^n \right].
\]

Now, observe that attribute values are independent across dimensions. So the probability that for every dimension \( k \), there is at least one observation \( x_i^k \in [-\tau_k, 0) \) and at least one observation \( x_j^k \in (0, \tau_k] \), is

\[
\prod_{k=1}^{m} \left( 1 - \left( \frac{1}{2} \right)^n \left[ (2 - \tau_k)^n + (2 - \bar{\tau}_k)^n - (2 - (\tau_k + \bar{\tau}_k))^n \right] \right).
\]

The desired probability is for the complement of this event, which yields the expression in the lemma.

The following functional form is used in the main text:

**Corollary 3.** In the special case in which the true function is \( f(x) = \mathbb{1}(x \in R) \) where \( R = [-a, a]^m \) for some \( a \in (0, 1) \), then:

\[
\overline{p}^n(a_i) = 1 - \left[ 1 - \left( 2 \left( \frac{2 - a}{2} \right)^n - (1 - a)^n \right) \right]^m.
\]
B Proofs for Main Results (Sections 5 and 6)

B.1 Preliminary Results

This section fixes an arbitrary set $\mathcal{C} \subseteq \Theta$ and action $a_i$, and provides conditions under which $a_i$ is strongly or weakly $\mathcal{C}$-rationalizable.

B.1.1 Sufficient Condition for Strong $\mathcal{C}$-Rationalizability

I first show that the property that $a_i$ can be rationalized using the same chain of best responses on $\mathcal{C}$, as defined in Definition 5, is sufficient for $a_i$ to be strongly $\mathcal{C}$-rationalizable.

**Lemma 5.** Action $a_i$ is strongly $\mathcal{C}$-rationalizable if $a_i$ can be rationalized under the same chain of best responses on $\mathcal{C}$.

**Proof.** Suppose action $a_i$ can be rationalized at every $\theta \in \mathcal{C}$ using the tuple $(R_j, (\nu[a_j])_{a_j \in R_j})_{j \in I}$. Then, at each $\theta \in \mathcal{C}$,

$$\int u_j(a_j, a_{-j}, \theta) d\nu[a_j](a_{-j}) \geq \int u_j(a_j', a_{-j}, \theta) d\nu[a_j](a_{-j}) \quad \forall a_j' \in A_j$$

Now consider an arbitrary distribution $Q \in \Delta(\Theta \times A_{-j})$ satisfying $\text{marg}_{\Theta} Q(\mathcal{C}) = 1$ and $Q(a_{-j} | \theta) = \nu[a_j](a_{-j})$ for every $\theta \in \Theta$ and $a_{-j} \in A_{-j}$. Then, the action $a_j$ is a best reply to the belief $Q$, since

$$\int u_j(a_j, a_{-j}, \theta) dQ(\theta, a_{-j}) = \int \int u_j(a_j, a_{-j}, \theta) dQ(a_{-j} | \theta) dQ(\theta)$$

$$= \int \int u_j(a_j, a_{-j}, \theta) d\nu[a_j] dQ(\theta)$$

$$\geq \int \int u_j(a_j', a_{-j}, \theta) d\nu[a_j] dQ(\theta)$$

$$= \int u_j(a_j', a_{-j}, \theta) dQ(\theta, a_{-j})$$

for every $a_j' \in A_j$, where the third line uses that $Q_\Theta := \text{marg}_{\Theta} Q$ assigns probability 1 to the set $\mathcal{C}$.

Now consider an arbitrary type $t_i \in T_i^\mathcal{C}$. By definition of the set $T_i^\mathcal{C}$, type $t_i$’s first-order belief $\text{marg}_{\Theta} \kappa_i(t_i)$ assigns probability 1 to $\mathcal{C}$. For each player $j$ and action $a_j \in R_j$, define $\pi[a_j] \in \Delta(\Theta \times T_{-j} \times A_{-j})$ to satisfy (1) $\text{marg}_{\Theta \times T_{-j}} \pi[a_j] = \kappa_j(t_j)$, and (2) $\pi[a_j](a_{-j} | \theta, t_{-j}) = \nu[a_j](a_{-j})$ for every $a_{-j} \in A_{-j}$, $\theta \in \Theta$, and $t_{-j} \in T_{-j}$. Then $Q := \text{marg}_{\Theta \times A_{-j}} \pi[a_j]$ has the properties described above—that is, $Q_\Theta(\mathcal{C}) = 1$ and $Q(a_{-j} | \theta) = \nu[a_j](a_{-j})$ for every $\theta$ and $a_{-j} \in A_{-j}$. So $a_i$ is a best reply to $Q$, and rationalizability of action $a_i$ player $i$ of type $t_i$ follows from the following proposition:
Proposition 2 (Dekel et al. (2007)). Fix any type profile \((t_j)_{j \in I}\). Consider any family of sets \(R_j \subseteq A_j\) such that every action \(a_j \in R_j\) is a best reply to a distribution \(\pi[a_j] \in \Delta(\Theta \times T_{-j} \times A_{-j})\) that satisfies \(\text{marg}_{\Theta \times T_{-j}} \pi[a_j] = \kappa_j(t_j)\) and \(\pi[a_j](a_{-j} \in R_{-j}[t_{-j}]) = 1\). Then, \(R_j \subseteq S_j^\epsilon[t_j]\) for every player \(j\).

Repeating this argument for every \(t_i \in T_i^\epsilon\), we have the desired conclusion that \(a_i\) is strongly \(\mathcal{C}\)-rationalizable. \(\square\)

B.1.2 Characterization of Weak \(\mathcal{C}\)-Rationalizability

The following lemma characterizes the sets \(\mathcal{C}\) for which an action \(a_i\) is weakly \(\mathcal{C}\)-rationalizable.

Lemma 6. Action \(a_i\) is weakly \(\mathcal{C}\)-rationalizable if and only if there exists a family \((R_j)_{j \in I}\) such that \(a_i \in R_i\), and also every \(a_j \in R_j\) is a best reply to a belief \(\nu[a_j] \in \Delta(\mathcal{C} \times R_{-j})\).

Proof. Suppose such a family \((R_j)_{j \in I}\) exists. Define a sequence of types \((\tilde{t}_j[a_j])_{j \in I, a_j \in R_j}\) where the first-order beliefs of these types satisfy

\[
\tilde{t}_j^1[a_j](\theta, t_{-j}) = \begin{cases} 
\nu[a_j](\theta, a_{-j}) & \text{if } t_{-j} = \tilde{t}_{-j}[a_{-j}] \\
0 & \text{otherwise}
\end{cases}
\]

Since for every player \(j\) and action \(a_j\), \(\text{marg}_\Theta \tilde{t}_j^1[a_j](\mathcal{C}) = \text{marg}_\Theta \nu[a_j](\mathcal{C}) = 1\), and additionally each type \(\tilde{t}_j[a_j]\) assigns positive probability only to other types in \((\tilde{t}_j[a_j])_{j \in I, a_j \in R_j}\), it follows that \(\tilde{t}_j[a_j] \in T_j^\epsilon\). Moreover, the action \(a_j\) is a best reply to the distribution \(\pi[a_j] \in \Delta(\Theta \times T_{-j} \times A_{-j})\) satisfying \(\text{marg}_{\Theta \times T_{-j}} \pi[a_j] = \kappa_j(\tilde{t}_j[a_j])\) and \(\text{marg}_{\Theta \times A_{-j}} \pi[a_j] = \nu[a_j]\). So, applying Proposition 2, \(a_i\) is rationalizable for player \(i\) with type \(\tilde{t}_i[a_i]\).

Conversely, suppose that \(a_i\) is rationalizable for player \(i\) of some type \(t_i \in T_i^\epsilon\). By definition of rationalizability, there exists a family of sets \(R_j[t_j] \subseteq A_j\) such that every action \(a_j \in R_j[t_j]\) is a best reply to a distribution \(\pi[a_j] \in \Delta(\Theta \times T_{-j} \times A_{-j})\) that satisfies \(\text{marg}_{\Theta \times T_{-j}} \pi = \kappa(t_j)\), and \(\pi(a_{-j} \in R_{-j}[t_{-j}]) = 1\). I will now show that each \(a_j \in R_j\) is a best reply to some \(\nu[a_j] \in \Delta(\mathcal{C} \times R_{-j})\). Since \(t_i \in T_i^\epsilon\) by assumption, \(\pi(t_{-j} \in T_{-j}^\epsilon) = 1\). For each player \(j\), choose an arbitrary type \(t_j\) in the support of the beliefs of type \(t_i\), and define \(\nu[a_j] = \text{marg}_{\Theta \times A_{-j}} \pi[a_j]\). Since the types in \(T_j^\epsilon\) have common certainty in \(\mathcal{C}\), it follows that \(\nu[a_j] \in \Delta(\mathcal{C} \times R_{-j})\) as desired. \(\square\)

B.2 Proof of Theorem 1

Throughout this proof, let the set of payoffs \(\Theta\) be endowed with the sup-norm.
(a) Suppose action $a_i$ is strictly rationalizable in the complete information game described by $\theta^\infty$. Then, there exists a family $(R_j)_{j \in I}$ of sets from $A_j$ where $a_i \in A_i$, and each action $a_j \in R_j$ is a strict best response to some distribution $\nu[a_j] \in \Delta(R_{-j})$; that is,

$$a_j = \arg\max_{a_j \in A_j} \int u_j(a_j', a_{-j}, \theta^\infty) d\nu[a_j] \quad \forall \ a_j \in R_j.$$ 

Since these are strict best replies, and action sets are finite, there must exist some $\delta > 0$ such that these actions are $\delta$-strict best replies; that is, for every player $j$ and action $a_j \in R_j$,

$$\int u_j(a_j, a_{-j}, \theta^\infty) d\nu[a_j] - \delta > \int u_j(a_j', a_{-j}, \theta^\infty) d\nu[a_j] \quad \forall a_j' \neq a_j.$$

By the assumption that payoffs are Lipschitz-continuous, there is a constant $K$ such that

$$\left| \int u_j(a_j, a_{-j}, \theta) d\nu[a_j] - \int u_j(a_j, a_{-j}, \theta^\infty) d\nu[a_j] \right| \leq K\|\theta - \theta^\infty\|_\infty.$$ 

So the tuple $(R_j, (\nu[a_j])_{a_j \in R_j})_{j \in I}$ rationalizes $a_i$ at every $\theta \in \{\theta^\infty\}^{\delta/K}$, the $\delta/K$-neighborhood of $\theta^\infty$. Applying Lemma 5, the action $a_i$ is strongly $C$-rationalizable for every $C \subseteq \{\theta^\infty\}^{\delta/K}$.

Finally, by Assumption 3,

$$\lim_{n \to \infty} \sup_{\mu \in M} \left| \mathbb{E}_{\mu}(Z_n)(\theta) - \theta^\infty \right| < \delta/K \quad \text{P-a.s.}$$ 

so the set of plausible parameters $\mathcal{C}(z_n)$ is almost surely contained in $\{\theta^\infty\}^{\delta/K}$ as $n \to \infty$. It directly follows that $p^n(a_i) \to 1$. Since $p^n(a_i) \leq \overline{p}(a_i)$ at every $n$, the desired statement follows.

(b) Suppose action $a_i$ is not rationalizable in the complete information game with payoffs $\theta^\infty$. By Assumption 3,

$$P^n(\mathcal{C}(z_n) \subseteq \{\theta^\infty\}^\epsilon) \to 1 \quad \forall \epsilon > 0$$

where $\{\theta^\infty\}^\epsilon$ to be the $\epsilon$-neighborhood of $\theta^\infty$ (in the sup-norm). So if we can show that for all $\epsilon$ sufficiently small, $\mathcal{C} \subseteq \{\theta^\infty\}^\epsilon$ implies that $a_i$ is not weakly $\mathcal{C}$-rationalizable, then it will follow that $\overline{p}(a_i) \to 0$.

Suppose towards contradiction that there exists a sequence of sets $\mathcal{C}^\epsilon \subseteq \{\theta^\infty\}^\epsilon$ where $a_i$ is weakly $\mathcal{C}^\epsilon$-rationalizable along this sequence. Then, applying Part (b) of Lemma 5, there must exist families $(R_j^\epsilon)_{j \in I}$ such that $a_i \in R_i^\epsilon$, and every $a_j \in R_j^\epsilon$ is a best reply to a belief $\nu^\epsilon[a_j] \in \Delta(\mathcal{C}^\epsilon \times R_{-j})$. Moreover, since action sets are finite, there is a finite number of
families of subsets of $A_j$. This implies existence of a family $(R_j)_{j \in \mathcal{I}}$ and subsequence $\epsilon_n \to 0$ along which each $(R_{j}^{\epsilon_n})_{j \in \mathcal{I}} = (R_{j})_{j \in \mathcal{I}}$.

For each player $j$ and action $a_j \in R_j$, define $BR_{a_j} : \Theta \to \Delta(A_{-j})$ to be the best reply correspondence mapping each game $\theta$ into the set of mixed profiles to which $a_j$ is a best reply. Recall that the best reply correspondence is upper hemi-continuous with respect to payoffs. Moreover, along the subsequence $\epsilon_n$, each $BR_{a_j}(\theta^{\epsilon_n})$ is nonempty, since it includes at least $\nu^{\epsilon_n}[a_j]$. Thus the set of mixed profiles to which $a_i$ is a best reply must also be nonempty at the limit $\theta^\infty$. So $(R_j)_{j \in \mathcal{I}}$ has the property that $a_i \in R_i$, and also that each $a_j \in R_j$ is a best reply to some distribution over $R_{-j}$ at the limiting payoffs $\theta^\infty$. Thus $a_i$ is rationalizable given payoffs $\theta^\infty$, yielding the desired contradiction.

B.3 Proof of Theorem 2

Since $a_i$ is strictly rationalizable for player $i$ in the game with payoffs $\theta^\infty$, it follows that $\delta^\infty > 0$. Following arguments similar to those used in Part (a) of the proof of Theorem 1, the action $a_i$ can be rationalized using the same chain of best responses on the $\delta^\infty/K$-neighborhood of $\theta^\infty$ (in the sup-norm). So, applying Part (a) of Lemma 5, if $\mathcal{C}(z_n) \subseteq \{\theta^\infty\}^{\delta^\infty/K}$, then the strategy $a_i$ is strongly $\mathcal{C}(z_n)$-rationalizable. This allows us to construct the lower bound

$$
\underline{P}(a_i) \geq Q^n(\{z_n : \mathcal{C}(z_n) \subseteq \{\theta^\infty\}^{\delta^\infty/K}\})
$$

$$
= Q^n \left( \left\{ z_n : \sup_{\mu \in \mathcal{M}} \| E_{\mu}(z_n)[\theta] - \theta^* \|_\infty \leq \delta^\infty/K \right\} \right)
$$

$$
\geq 1 - \frac{K}{\delta^\infty} \mathbb{E} \left( \sup_{\mu \in \mathcal{M}} \| E_{\mu}(z_n)[\theta] - \theta^\infty \|_\infty \right)
$$

using Markov’s inequality in the final line.

C Proofs for Special Sets of Learning Rules

C.1 Proof of Claim 3

Using standard formulas for Bayesian updating, the expected value of $v$ under learning rule $\mu_{\pi,q}$ given data $z_n$ is

$$
\frac{1}{n} \left( 1 + \frac{1 - \pi}{\pi} \left( \frac{1 - q}{q} \right)^{n(2z_n-1)} \right)
$$

(7)

where $z_n = \frac{1}{n} \sum_{n'=1}^{n} z_{n'}$ is the average realization in the sequence $z_n$.  

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Suppose without loss that the parameter value is $v = 1$, and let $q^* \in (1/2, 1)$ be the true frequency of $z = 1$. By the strong Law of Large numbers, there is a measure one set of sequences $z$ satisfying $\lim_{n \to \infty} \left( \frac{1}{n} \sum_{n'=1}^{n} z_{n'} \right) = q^*$. The expression in (7) converges to 1 on this set for every learning rule $\mu_{\pi,q} \in \mathcal{M}$. So Assumption 2 is satisfied, and the limiting game is the one where players have common certainty that $v = 1$. Since entering is not rationalizable in this game, it follows that $p_{p8q} = 0$.

I will show that despite this, the probability $p_{n} \to 1$ as $n \to \infty$. Fix an arbitrary $n$, and condition on the event $\{ \overline{z}_n > 1/2 \}$. Then $2\overline{z}_n - 1 > 0$, implying that $((1-q)/q)^{n(2\overline{z}_n-1)}$ is bounded between $1/2$ and $1$ on the domain $q \in (1/2, 1)$, while the image of $(1-\pi)/\pi$ is all of $\mathbb{R}_+$. So the display in (7) ranges from zero to 1; that is,

$$C(z_n) = \left\{ \frac{1}{1 + \frac{1 - \pi}{\pi} \left( \frac{1 - q}{q} \right)^{n(2\overline{z}_n-1)}} : \pi \in (0,1), q \in (1/2,1) \right\} = (0,1)$$

and there exist values $\underline{p}, \overline{p} \in C(z_n)$ satisfying $\underline{p} < p < \overline{p}$. Entering is rationalizable when the Seller believes that $v = \underline{p}$ with probability 1, and believes with probability 1 that the Buyer believes $v = \underline{p}$ with probability 1.

Again by the law of large numbers, the measure of datasets with majority realizations of $z = 1$ converges to 1 as $n \to \infty$; that is, $P^n(\{z_n : z_n > 1/2\}) \to 1$. Since $p^n(a_i)$ is at least $P^n(\{z_n : z_n > 1/2\})$, it immediately follows that $\lim_{n \to \infty} p^n(a_i) = 1$, as desired.

### C.2 Proof of Corollary 1

First observe that $\delta^\infty = \beta - 1$, since the action $\text{Strict}$ is $\delta$-strictly rationalizable for every $\delta < \beta - 1$ and not for any $\delta \geq \beta - 1$. It remains to determine $\mathbb{E} \left[ \sup_{\theta' \in \mathcal{C}(Z_n)} \| \theta' - \theta^\infty \|_2 \right]$. Write $\overline{Z}_n$ for the (random) empirical mean of $n$ signal realizations. Then, using standard formulas for updating to Gaussian signals:

$$\mathbb{E} \left( d_H(\mathcal{C}(Z_n), \theta^\infty) \right) = \mathbb{E} \left[ \max_{x \in [-\eta,\eta]} \left( \left| \beta - x + n\overline{Z}_n \right| \right) \right]$$
We can further bound the RHS as follows:

\[
E \left[ \max_{x \in [-\eta, \eta]} \left( \beta - \frac{x + nZ_n}{n+1} \right) \right] \leq E \left( \left| \beta - \frac{nZ_n}{n+1} \right| \right) + \max_{x \in [-\eta, \eta]} \left| \frac{x}{n+1} \right|
\]

\[
= E \left( \left| \beta - \frac{nZ_n}{n+1} \right| \right) + \eta/(n+1)
\]

\[
\leq E \left( \left| \beta - Z_n \right| \right) + \eta/(n+1)
\]

\[
= \sqrt{\frac{2}{n\pi}} + \frac{\beta + \eta}{n+1}
\]

using in the final line the expected absolute deviation of the empirical mean of \( n \) observations from a Gaussian distribution (Geary, 1935). Finally, the map \( g : \Theta \to U \) has Lipschitz constant 1. Applying Theorem 2, we have the desired bound.

C.3 Proof of Corollary 2

First consider arbitrary \( \pi, \bar{\pi}, q, \bar{q} \) satisfying \( 0 < \pi < \bar{\pi} < 1 \) and \( 1/2 < \bar{q} < \bar{q} < 1 \), and consider the set of learning rules \( \mathcal{M} \) identified with \( (\pi, q) \in [\bar{\pi}, \bar{\pi}] \times [q, \bar{q}] \). Fix a pair \( (\pi, q) \in \mathcal{M} \), and a data sequence \( z_n \) with average realization \( \bar{z}_n > 1/2 \). The expected value of \( v \) under the belief \( \mu_{\pi, q}(z_n) \) is given in (7); henceforth call this expression \( \hat{v}(\pi, q, z_n) \). Entering is rationalizable for a Seller with common certainty in \( \mathcal{C}(z_n) \) if and only if there exist \( \pi, \pi' \in [\bar{\pi}, \bar{\pi}] \) and \( q, q' \in [q, \bar{q}] \) satisfying

\[
\hat{v}(\pi, q, z_n) < \pi < \hat{v}(\pi', q', z_n).
\]  

Let \( Z_n^* \) be the set of all sequences \( z_n \) satisfying (8).

Since the state space is binary, each empirical measure \( \hat{Q}(z_n) \in \Delta(\{0, 1\}) \) can be identified with its average realization \( \bar{z}_n \), which is also the probability assigned to \( z = 1 \). The KL-distance between \( \hat{Q}(z_n) \) and the actual signal-generating distribution \( Q = (q^*, 1-q^*) \) is

\[
D_{KL}(\hat{Q}(z_n) \mid Q) = q^* \log \left( \frac{q^*}{\bar{z}_n} \right) + (1-q^*) \log \left( \frac{1-q^*}{1-\bar{z}_n} \right)
\]

and this expression is monotonically increasing in \( |\bar{z}_n - q^*| \). Thus, we seek the value of \( \bar{z}_n \) closest to \( q^* \) for which (8) is satisfied.

Suppose \( \bar{z}_n > 1/2 \). Since by assumption \( \bar{\pi} > \pi \) and \( \bar{q} > 1/2 \), it must be that \( \hat{v}(\bar{\pi}, \bar{q}, z_n) > \pi \).

It remains to determine when \( \hat{v}(\pi, q, z_n) < \pi \) is satisfied for some other \( (\pi, q) \in \mathcal{M} \). Since
\( \hat{v}(\pi, q, z_n) \) is monotonically decreasing in both \( \pi \) and \( q \) for sequences \( z_n \) satisfying \( z_n > 1/2 \), a sufficient condition is \( \hat{v}(\pi, q, z_n) < p \). Using (7), this inequality requires

\[
1 \left( 1 + \frac{1 - \pi}{\pi} \left( \frac{1 - q}{q} \right)^{n(2z_n-1)} \right) < p
\]

which can be rewritten

\[
\begin{align*}
\hat{z}_n & \leq \frac{1}{2} \left( 1 + \frac{1}{n} \log(1-\hat{q}/\hat{q}) \left( \frac{\pi}{1 - \pi} \cdot \frac{1 - p}{p} \right) \right) := \hat{z}^*_n.
\end{align*}
\]

Since \( \hat{z}^*_n \cdot n \) need not be an integer, the empirical measure \( (\hat{z}^*_n, 1 - \hat{z}^*_n) \) may not be achievable by any empirical measure \( \hat{Q}_n \) for finite \( n \). Thus, \( Q^*_n \) is instead given by \( ([z^*_n \cdot n] / n, 1 - ([z^*_n \cdot n] / n) \), and

\[
D_{KL}(Q^*_n || Q) = q^* \log \left( \frac{q^*}{[z^*_n \cdot n] / n} \right) + (1 - q^*) \log \left( \frac{1 - q^*}{1 - [z^*_n \cdot n] / n} \right)
\]

Plugging in the given parameter values, and applying Proposition 4, yields the expression in the corollary.

### C.4 Proof of Proposition 1

Enumerate the learning rules in \( \mathcal{M} \) by \( \mu_1, \ldots, \mu_K \). For every \( k = 1, \ldots, K \), define

\[
X^n_k = 1 \left( \mathbb{E}_{\mu_\pi(Z_n)}(\theta) \notin \Theta^n \right)
\]

where where \( \Theta^n \) is the set as defined in Definition 4. Write \( G^n_k \) for the distribution of random variable \( X^n_k \), and \( G^n \) for the joint distribution of random variables \( (X^n_k)_{k=1}^K \), noting that \( p^n_{\mu_k}(a_i) = G^n_k(0) \). By Sklar’s theorem, there exists a copula \( C : [0, 1]^K \to [0, 1] \) such that

\[
G^n(x_1, \ldots, x_K) = C \left( G^n_1(x_1), \ldots, G^n_K(x_K) \right)
\]

for every \( (x_1, \ldots, x_K) \in \mathbb{R}^K \). Using the Frechet-Hoeffding bound,

\[
1 - K + \sum_{k=1}^K G^n_k(x_k) \leq C \left( G^n_1(x_1), \ldots, G^n_K(x_K) \right) \leq \min_{k \in \{1, \ldots, K\}} G^n_k(x_k).
\]

Under Assumption 6, \( p^n_M \) is identical to the probability that the set of expected parameters is contained within \( \Theta^n \), and hence \( p^n_M(a_i) = G^n(0, \ldots, 0) \). Thus:

\[
1 - K + \sum_{\mu \in \mathcal{M}} p^n_{\mu}(a_i) \leq p^n_M(a_i) \leq \min_{\mu \in \mathcal{M}} p^n_{\mu}(a_i).
\]

as desired.
Online Appendix

D Additional Material Improving Theorem 1

Part (a) of Theorem 1 provides a sufficient condition for the confidence set \([\bar{p}^n(a_i), \bar{p}^n(a_i)]\) to converge to certainty—\(a_i\) is strictly rationalizable at \(\theta^\infty\)—and Part (b) of Theorem 1 provides a necessary condition—\(a_i\) is rationalizable at \(\theta^\infty\). The condition that \(a_i\) is strictly rationalizable is not necessary, as I demonstrate in Section D.2, and the condition that \(a_i\) is rationalizable is not sufficient, as I demonstrate in Section D.3.2. The purpose of this appendix is to explain the nature of this gap. While a complete characterization is beyond the pursuit of this paper, in Section D.4, I provide a weaker necessary condition—weak strict-rationalizability—that makes the gap smaller.

D.1 Overview

The main takeaways from this section are summarized in Figure 5. Suppose \(a_i\) is strictly rationalizable at \(\theta^\infty\); then, the analyst’s confidence interval for \(a_i\) must converge to certainty. A simple necessary condition for this convergence to occur is that \(a_i\) is rationalizable at \(\theta^\infty\), and a stronger necessary condition is that \(a_i\) is weakly-strict rationalizable, as defined in Section D.4. Again, this property is not sufficient. Examples are given to explain each of the gaps.

![Diagram showing the relationship between strict rationalizability, weak-strict rationalizability, and rationalizability](image)

Figure 5: The shaded region depicts limiting parameters \(\theta^\infty\) for which the analyst’s confidence set necessarily converges to certainty.

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26See Chen and Takahashi (2017) for concurrent work that makes progress towards this goal.
D.2 Strict Rationalizability is Not Necessary

Consider the following complete information game

<table>
<thead>
<tr>
<th></th>
<th>$a_3$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$\theta,0$</td>
<td>$\theta,0$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$0,0$</td>
<td>$0,0$</td>
</tr>
</tbody>
</table>

and suppose that the limiting value of the parameter is $\theta^\infty = 1$. Then, the action $a_1$ is strictly dominant in the limiting game, and also in all games with nearby parameter values. It is straightforward to show that the action $a_1$ is rationalizable given common certainty of a small enough neighborhood around $\theta^\infty$, and so Assumption 3 implies $\lim_{n \to \infty} [p^n(a_i),p^n(a_i)] = \{1\}$. But action $a_1$ is not strictly rationalizable in the original game.\(^{27}\)

D.3 Rationalizability is not Sufficient

I show next that rationalizability of $a_i$ at $\theta^\infty$ is not sufficient for the analyst’s confidence set for $a_i$ to converge to certainty. Section D.3.1 provides a simple example to this effect: if $a_i$ is on the boundary of $\Theta^{a_i}$, then common certainty of shrinking neighborhoods around $\theta^\infty$ does not guarantee rationalizability of $a_i$. More surprisingly, common certainty in arbitrarily small open sets within the interior of $\Theta^{a_i}$ also does not guarantee rationalizability of $a_i$, and I provide an example of this in Section D.3.2.

D.3.1 $\theta^\infty$ is on the Boundary of $\Theta^{a_i}$

Consider the following two-player game, parametrized by $\theta \in \mathbb{R}$:

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$\theta,\theta$</td>
<td>$0,0$</td>
</tr>
<tr>
<td>$b$</td>
<td>$0,0$</td>
<td>$1,1$</td>
</tr>
</tbody>
</table>

Suppose that the limiting parameter $\theta^\infty = 0$, so that $a$ is rationalizable in the limiting game, but not strictly rationalizable. It is straightforward to see that common certainty of shrinking neighborhoods of $\theta^\infty$ does not guarantee rationalizability of $\theta$, as the type with common certainty of any $\theta' < 0$ considers $a$ to be strictly dominated.

\(^{27}\)It is not possible to construct a family of sets $(R_1,R_2) \subseteq A_1 \times A_2$ with the property that each action $a_j \in R_j$ is a strict best reply to a distribution over $R_{-j}$.
D.3.2  $\theta^{\infty}$ is in the Interior of $\Theta^{a_i}$

But even if $\theta^{\infty}$ is not on the boundary of the set $\Theta^{a_i}$, it may be that common certainty of a shrinking neighborhood of $\theta^{\infty}$ does not guarantee rationalizability of $a_i$. Consider the following four-player game. Players 1 and 2 choose between actions in \{a, b\}, and player 3 chooses between matrices from \{l, r\}. Their payoffs are:

\[
\begin{array}{cc|cc}
& a & b & a & b \\
\hline
a & 1,1,0 & 0,0,0 & a & 0,0,0 & 0,0,0 \\
b & 0,0,0 & 0,0,0 & b & 0,0,0 & 1,1,0 \\
\end{array}
\]

(9)

A fourth player predicts whether players 1 and 2 chose matching actions or mis-matching actions. He receives a payoff of 1 if he predicts correctly (and 0 otherwise).\(^{28}\) Player 4’s action does not affect the payoffs of the other three players.

Let the state space $\Theta = \mathbb{R}^{64}$ be the set of all payoff matrices given these actions, where the payoffs described above are a particular $\theta$. Match is clearly rationalizable for player 4 at $\theta$; it is also rationalizable for player 4 on a neighborhood of $\theta$ (in the Euclidean metric).\(^{29}\)

Nevertheless, I will show existence of a sequence of types for player 4 with common certainty in increasingly small neighborhoods of $\theta$, given which Match fails to be rationalizable. Along this sequence, player 4 believes that $a$ is uniquely rationalizable for player 1, while $b$ is uniquely rationalizable for player 2, so the action Match is strictly dominated.

Define $\theta^{1}_\varepsilon$ to be the following perturbation of the payoff matrix $\theta$ (with player 4’s payoffs unchanged):

\[
\begin{array}{cc|cc}
& a & b & a & b \\
\hline
a & 1,1,0 & 0,0,0 & a & 0,0,-\varepsilon & 0,0,-\varepsilon \\
b & 0,0,0 & -\varepsilon,0,0 & b & 0,0,-\varepsilon & 1,1,-\varepsilon \\
\end{array}
\]

(10)

Let $\theta^{2}_\varepsilon$ correspond to the following payoff matrix (again with player 4’s payoffs unchanged):

\[
\begin{array}{cc|cc}
& a & b & a & b \\
\hline
a & 1,1,-\varepsilon & 0,0,-\varepsilon & a & -\varepsilon,0,0 & 0,0,0 \\
b & 0,0,-\varepsilon & 0,0,-\varepsilon & b & 0,0,0 & 1,1,0 \\
\end{array}
\]

(11)

\(^{28}\)In more detail: player 4 chooses between \{Match, Mismatch\}. His payoff from Match is 1 if players 1 and 2 choose the same action (both $a$ or both $b$) and 0 otherwise; his payoff from Mismatch is 1 if players 1 and 2 chose different actions ($a$ and $b$ or flipped), and 0 otherwise.

\(^{29}\)Suppose neither $l$ nor $r$ are strictly dominated for player 1; then, all actions are rationalizable for player 1-3, so Match is rationalizable for player 4. If either $l$ or $r$ is strictly dominated for player 1, then one of the following will be a rationalizable family: \{l\} × \{a\} × \{a\} × \{Match\}, \{l\} × \{a, b\} × \{a, b\} × \{Match\}, \{r\} × \{b\} × \{b\} × \{Match\}, or \{r\} × \{a, b\} × \{a, b\} × \{Match\}. Thus, Match is rationalizable for player 4.
Let \( \varepsilon > 0 \). If player 1 has common certainty in the state \( \theta_1^\varepsilon \), then \( a \) is his uniquely rationalizable action: \( l \) strictly dominates \( r \) for player 3, given which \( a \) strictly dominates \( b \) for player 1. By a similar argument, if player 2 has common certainty in the state \( \theta_2^\varepsilon \), then \( b \) is his uniquely rationalizable action. These statements hold for \( \varepsilon \) arbitrarily small. Construct a sequence of types \( (t_4^n) \) for player 4, where each type \( t_4^n \) has common certainty that player 1 has common certainty in the state \( \theta_1^{t_4^n} \) and player 2 has common certainty in the state \( \theta_2^{t_4^n} \). Then, player 4 of type \( t_4^n \) has common certainty in an \( \varepsilon \)-neighborhood of \( \theta \), but only one rationalizable action: Mismatch. Take \( \varepsilon_n \to 0 \) (with each \( \varepsilon_n > 0 \)) and the desired conclusion obtain: rationalizability of Match holds at \( \lim_{n \to \infty} \varepsilon_n \) but fails to hold arbitrarily far out along the sequence \( \varepsilon_n \).

### D.4 Weaker Necessary Condition for Strong \( \mathcal{C} \)-Rationalizability

As described in Section 3, strict rationalizability can be defined as the limit of a process of iterative elimination of actions that are never a strict best reply. This procedure is known to be sensitive to the manner of elimination. Consider specifically all the orders of elimination in which \( \text{at most one} \) action is eliminated at a time. Formally, define \( W_1^1 := \Delta_i \) for every player \( i \). Then, for each \( k \geq 2 \), recursively remove (at most) one action in \( W_i^k \) that is not a strict best reply to any opponent strategy \( \alpha_{\cdot \cdot} \in \Delta(W_{i-1}^{k-1}) \). Let

\[
W_i^\infty = \bigcap_{k \geq 1} W_i^k
\]

be the set of player \( i \) actions that survive every round of elimination, and define \( \mathcal{W}_i^\infty \) to be the intersection of all sets \( W_i^\infty \) that can be constructed in this way.

**Definition 6.** Say that an action \( a_i \) is weakly strict-rationalizable if \( a_i \in \mathcal{W}_i^\infty \).

In the game introduced in Section D.2, there are two patterns of one-at-a-time elimination. One possibility is

\[
\begin{align*}
(a_3 & a_4) \\
(a_1 & 1,0 \rightarrow a_1 1,0) \\
(a_2 & 0,0 \rightarrow a_2)
\end{align*}
\]

in which action \( a_2 \) is eliminated for player 1 and action \( a_4 \) is eliminated for player 2, so that actions \( a_1 \) and \( a_3 \) remain. Another possibility is

\[
\begin{align*}
(a_3 & a_4) \\
(a_1 & 1,0 \rightarrow a_1 1,0) \\
(a_2 & 0,0 \rightarrow a_2)
\end{align*}
\]
in which action $a_2$ is eliminated for player 1 and action $a_3$ is eliminated for player 2, so that actions $a_1$ and $a_4$ remain. The action $a_1$ survives both procedures; hence, it is weakly strict-rationalizable.

I show next that weak strict-rationalizability at the limiting game $\theta^\infty$ is a necessary condition for the confidence set to converge to certainty:

**Proposition 3.** $\lim_{n \to \infty} [p^n(a_i), p^n(a_i)] = \{1\}$ only if $a_i$ is weakly-strict rationalizable in the limiting game $\theta^\infty$.

This proposition follows from the following lemma, which may be of independent interest. The lemma says that weak-strict rationalizability characterizes the interior of $\Theta^{a_i}$; that is, action $a_i$ is weakly-strict rationalizable in every game $\theta$ in the interior of $\Theta^{a_i}$, and if $\theta$ is in the interior of $\Theta^{a_i}$, then $a_i$ must be weakly-strict rationalizable in the game $\theta$.

**Lemma 7.** $\theta \in \text{Int} (\Theta^{a_i})$ if and only if $a_i$ is weakly strict-rationalizable in the complete information game indexed to $\theta$.

**Proof.** If: Suppose the game $\theta$ is not in the interior of $\Theta^{a_i}$. There must then exist a sequence $\theta^n \to \theta$ (converging in the sup-norm), where for large $n$, the game $\theta^n$ is also not in the interior of $\Theta^{a_i}$. Thus in each late game $\theta^n$, there is an order of elimination of strictly dominated strategies that removes $a_i$. Moreover, since action sets are finite, there is a finite number of possible such orders of elimination. This implies existence of a subsequence along which the same order of iterated elimination of strategies removes $a_i$. At the limiting payoffs $\theta$, action $a_i$ must fail to survive elimination of weakly dominated strategies along this order, and is therefore not weakly strict-rationalizable.

Only if: Suppose $a_i$ is not weakly strict-rationalizable. Then, there exists a sequence of sets $(W^k_j)_{k \geq 1}$ for every player $j$ satisfying the recursive description, such that $a_i \notin W^K_j$ for some $K < \infty$. To show that $\theta$ is not in the interior of $\Theta^{a_i}$, I construct a sequence of payoff functions $\theta^n$ with $\theta^n \to \theta$ such that $a_i$ is not rationalizable in any late game along this sequence.

For every $n \geq 1$, recursively define $\theta^n$ according to the following procedure. First, for every player $j$, let $\theta^{n,0} = \theta$. Then, for every $l \geq 1$, define $\theta^{n,l}$ such that

$$u_j(a_j, a_{-j}, \theta^{n,l}) = \begin{cases} u_j(a_j, a_{-j}, \theta^{n,l-1}) + \epsilon/n & \forall \ a_j \in W^l_j, a_{-j} \in A_{-j} \\ u_j(a_j, a_{-j}, \theta^{n,l-1}) & \forall \ a_j \notin W^l_j, a_{-j} \in A_{-j} \end{cases}$$

That is, we iteratively increase the payoffs of the surviving strategies at each round of elimination (according to $(W^k_j)_{k \geq 1}$) by $\epsilon/n$. Finally, let $\theta^n = \theta^{n,K}$.

I claim that $a_i$ is not rationalizable in any complete information game $\theta^n$, when $n$ is sufficiently large. In game $\theta^n$, let $S^{k,n}_j$ be the set of player $j$ actions that survive $k$ rounds
of iterated elimination of strictly dominated strategies. I will show that in games $\theta^n$ where $n$ is sufficiently large, the sets $S_j^{k,n} = W_j^k$ for all $k$ and every player $j$.

Proceed by induction. Trivially, $S_j^{0,n} = W_j^0 = A_j$ for every $j$ and $n$. Suppose $S_j^{k,n} = W_j^k$ for every player $j$ and round $k \leq L$ when $n$ is sufficiently large. Now consider any action $a_j \in S_j^{L,n}$. Suppose $a_j$ is a strict best response to some strategy $\alpha_{-j} \in \Delta(W_j^L)$; then clearly, $a_j \in W_j^{L+1}$. The claim below implies that also $a_j \in S_j^{L+1,n}$ for $n$ sufficiently large.

**Claim 5.** Suppose $a_j$ is a strict best response to some strategy $\alpha_{-j} \in \Delta(A_{-j})$ and define

$$\gamma = \frac{1}{2}|u_j(a_j, \alpha_{-j}, \theta) - \max_{a_j' \neq a_j} u_j(a_j', \alpha_{-j}, \theta)| > 0$$

Set $N > \frac{\epsilon K}{\gamma}$. Then, action $a_j$ is a strict best response to $\alpha_{-j}$ in every game $\theta^n$ when $n > N$.

**Proof.** Define $a_j^* = \arg\max_{a_j' \neq a_j} u(a_j', \alpha_{-j}, \theta^n)$. Then,

$$u_j(a_j, \alpha_{-j}, \theta^n) - u_j(a_j^*, \alpha_{-j}, \theta^n) = u_j(a_j, \alpha_{-j}, \theta^n) - u_j(a_j, \alpha_{-j}, \theta) + u_j(a_j, \alpha_{-j}, \theta) - u_j(a_j^*, \alpha_{-j}, \theta) + u_j(a_j^*, \alpha_{-j}, \theta) - u_j(a_j^*, \alpha_{-j}, \theta^n) \geq -(\epsilon K)/n + 2\gamma - (\epsilon K)/n \geq 2\gamma - (2\epsilon K)/N > 0$$

using in the penultimate inequality that $n > N$, and in the final inequality that $N > \frac{\epsilon K}{\gamma}$. □

Thus we have $W_j^{L+1} = S_j^{L+1,n}$ for large $n$ in this case.

Suppose that $a_j$ is only a weak best response to some strategy $\alpha_{-j} \in \Delta(W_j^L)$. In this case, there are two possibilities: (1) If $a_j \in W_j^{L+1}$, then by construction of $\theta^n$, action $a_j$ is rendered a strict best response to $\alpha_{-j}$ under $\theta^n$; thus, $a_j \in S_j^{L+1,n}$ for all $n$ sufficiently large. (2) Otherwise, if $a_j \notin W_j^{L+1}$, then there exists some alternative action $a_j'$ satisfying $u_j(a_j', \alpha_{-j}, \theta) = u_j(a_j, \alpha_{-j}, \theta)$ and also $a_j' \in W_j^{L+1}$. By construction of payoffs $\theta^n$, the action $a_j'$ yields a strictly higher payoff than $a_j$ against $\alpha_{-j}$ in all games $\theta^n$. Repeating this argument for any $\alpha_{-j} \in \Delta(W_j^L)$ to which $a_j$ is a weak best reply, we have that $a_j \notin S_j^{L+1,n}$ for $n$ sufficiently large (using Claim 5). Finally, if $a_j$ is not a weak best response to any strategy $\alpha_{-j} \in \Delta(W_j^L)$, then both $a_j \notin W_j^{L+1}$ and also $a_j \notin S_j^{L+1,n}$ for $n$ sufficiently large.

Therefore $S_j^{k,n} = W_j^k$ for every $k$ and $n$ sufficiently large. Since $a_j \notin W_j^K$, also $a_j \notin S_j^{\infty,n}$ for $n$ sufficiently large, as desired. Finally, by construction $\theta^n \to \theta$, so $\theta \notin \text{Int}(\Theta^n)$, as desired. □