Games of Incomplete Information
Played by Statisticians

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Abstract

This paper proposes a foundation for heterogeneous beliefs in games, in which
disagreement arises not because players observe different information, but because
they learn from common information in different ways. Players may be misspecified,
and may moreover be misspecified about how others learn. The key assumption is
that players nevertheless have some common understanding of how to interpret the
data; formally, players have common certainty in the predictions of a class of learning
rules. The common prior assumption is nested as the special case in which this class
is a singleton. The main results characterize which rationalizable actions and Nash
equilibrain can be predicted when agents observe a finite quantity of data, and how
much data is needed to predict various solutions. This number of observations needed
depends on the degree of strictness of the solution and speed of common learning.

1 Introduction

How are beliefs formed? And how do individuals come to form beliefs over the beliefs
of others? Predictions of play in incomplete information games depend crucially on our
answers to these questions. The classic approach gives players a common prior over states
of the world, and assumes that they use Bayesian updating to form a posterior belief given
new information.1 But this approach imposes strong restrictions on the extent to which

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1The related, stronger, notion of rational expectations assumes moreover that this common prior distribu-
tion is in fact the “true” distribution shared by the modeler.
players can disagree: for example, it implies that beliefs that are commonly known are identical (Aumann, 1976), and that repeated communication of beliefs will eventually lead to agreement (Geanakoplos and Polemarchakis, 1982). These implications are hard to believe when taken literally: they conflict with considerable empirical evidence that individuals publicly disagree, and also with our basic, day-to-day, experience that individuals interpret information in different ways. Moreover, there are conflicts between the predictions of the common prior theory and certain observed economic behaviors—for example, the well known result that the common prior assumption precludes speculative trade (Milgrom and Stokey, 1982).

The main contribution of this paper is to propose a simple framework that relaxes the common prior assumption in a structured way. The proposed approach takes a statistical view of belief formation: economic actors observe a sequence of data (generated via a stochastic process) and extrapolate from that data to predict payoff-relevant unknowns. For example, investors forecast future stock returns based on past returns. Theoretically, given infinite data about a stationary environment, agents can learn a complete theory of the relationship between observables and outcomes. But when data is “partial” (for example, outcomes are observed for settings that are similar but not identical to the current environment) and limited in quantity, there can be many plausible and competing viewpoints on what the data implies.

The proposed framework thus permits “ambiguity” in how to interpret the data, formalized as a class of rules for learning from data, which structure the potential disagreement. I define a learning rule to be any map from data into beliefs over payoffs. For example, the common prior assumption can be described by a learning rule that Bayesian updates from a specific (and common) view of the relationship between data and payoff-relevant unknowns. In general, I allow for a set of learning rules (potentially misspecified). The premise of this approach is that even in settings where there is not a single accepted way to interpret data, there is often domain knowledge about a class of reasonable approaches. For example, in the forecasting example above, there may be basic competing theories for how the returns are generated, i.e. corresponding to different parametrized families of return processes. Alternatively, players may take different averages of the historical data, based on different assumptions about which time periods are most relevant.

Thus, the standard approach is enriched with two new primitives: a class of learning rules.

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2 In financial markets, players publicly disagree in their interpretations of earnings announcements (Kandel and Pearson, 1995), valuations of financial assets (Carlin et al., 2013), forecasts for inflation (Mankiw et al., 2004), forecasts for stock movements (Yu, 2011), and forecasts for mortgage loan prepayment speeds (Carlin et al., 2014). players publicly disagree also in matters of politics (Wiegel, 2009) and climate change (Marlon et al., 2013).

3 See for example case-based learning (Gilboa and Schmeidler, 1995; Gilboa et al., 2008).
and a *data-generating process*. Given any data set, the class of learning rules determines a plausible range of disagreement about payoffs. A final assumption, *Common Inference*, structures the approach: for every realization of the data, players have common certainty in the beliefs induced by the class of learning rules.\textsuperscript{4} Thus, although players can disagree, and even have common knowledge of disagreement, the extent of their disagreement is constrained.

Section 5 characterizes the conditions on the class of learning rules that guarantee common learning (Cripps et al., 2008).\textsuperscript{5} It is necessary and sufficient that beliefs induced by different learning rules weakly converge to a limiting belief, and that this convergence occurs *uniformly* over the set of learning rules. Loosely, this requires that the set of learning rules is not too large. The case in which players commonly learn receives special emphasis in this paper, and I refer to the limiting belief as a *limiting common prior*.\textsuperscript{6}

Sections 6 and 7 focus on the restrictions on strategic behavior that are imposed by a class of learning rules, a data-generating process, and Common Inference. The key object of study is the probability (with respect to the data generating process) that a strategic prediction will hold under Common Inference. Specifically, I consider the probability that the prediction holds uniformly over all types consistent with Common Inference (call these predictions *robust*), and also the probability that the prediction holds for some type consistent with Common Inference (call these predictions *plausible*). Informally, the larger the two probabilities are, the more confidence we should have in the corresponding predictions. In this way, the probabilities serve as different *continuous* metrics for the strength of a prediction. Both measures are indexed to the number of observations, so that the strength of a prediction varies depending on how many observations players have seen.

The case in which players commonly observe a large quantity of data is considered in Section 6. The main takeaway is that when there is a limiting common prior, then strategic predictions that hold strictly given infinite data can also be made (with high probability) when players observe a large finite quantity of data. Formally, under the assumption of a limiting common prior, the probability that an action profile is a plausible equilibrium converges to 1 (as the number of observations grows large) if and only if it is an equilibrium under the limiting common prior; the probability that an action profile is a robust equilibrium converges to 1 if and only if it is a *strict* equilibrium under the limiting common prior. Similar statements hold for rationalizability, although there are additional subtleties in the appro-

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\textsuperscript{4}That is, all players assign probability 1 to this set of beliefs, believe with probability 1 that all other players assign probability 1 to this set of beliefs, and so forth.

\textsuperscript{5}That is, players’ own beliefs converge to this distribution, they believe with high probability that all other players’ beliefs converge to this distribution, and so forth.

\textsuperscript{6}One can interpret this limit as the point at which learning has removed all differences in beliefs that are not due to differences of information. From this point forward, players indeed share a common perception of the world.
appropriate notion of “strictness”: I introduce a new definition for “weakly” strict-rationalizable actions, which is a necessary condition for rationalizable predictions to be robust with high probability when the number of observations is large (the standard concept of strict rationalizability is a sufficient condition).

Finally, actions are plausibly rationalizable with high probability when the number of observations is large if and only if they are rationalizable under the limiting common prior. Taken together, these results imply that the simplifying assumption of a common prior is largely without loss if we believe that players observe a large amount of data, and there is a limiting common prior.

But when the number of observations is limited (the more practically relevant setting), then play can be quite different from the limit game. In Section 7, I provide a lower bound for the probability that a strategic prediction is robust, when the number of observations is some arbitrary (and potentially small) $n$. This bound depends on two key properties: (a) First, it is increasing in the speed at which different learning rules jointly learn the true value of the parameter. Thus, the more “complex” the learning problem, and the larger the set of learning rules, the lower the probability. (b) Second, the bound is increasing in a cardinal measure of strictness of the solution that I define. Say that an action profile is a $\delta$-strict $NE$ if each player’s prescribed action is at least $\delta$ better than his next best action; and say that an action profile is $\delta$-strict rationalizable if it can be rationalized by a chain of best responses, in which each action yields at least $\delta$ over the next best alternative. This parameter $\delta$ turns out to determine how much estimation error the solution can withstand—the lower the degree of strictness (the smaller the parameter $\delta$), the slower convergence is.

These bounds show that when players form beliefs from data using different learning rules, then new channels—in particular, the amount of common knowledge over how to interpret data, and the “dimensionality” or “complexity” of the learning problem—emerge as determinants of strategic behavior.

Section 8 explores the case in which the data-generating process is not exogenously determined, but can be influenced by an external actor—for example, the federal reserve board decides what data to release about various financial and macroeconomic indicators. How might a designer be able to manipulate behavior either by choosing the nature of public information? I present an example using the proposed framework, in which the provision of extraneous public information deters coordination.

Section 9 examines modeling choices made in the main text and discusses the extent to which these can be relaxed—specifically, by allowing for an “approximate” limiting common prior, and allowing for i.i.d. private data.

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7 The difference between these two notions of strict rationalizability regards the order of elimination, and is of independent interest.

8 This reverses the more familiar concept of $\epsilon$-equilibrium, which requires that each player’s prescribed action is no more than $\epsilon$ worse than the best action.
Section 10 surveys the related literature. This paper primarily builds on a literature that studies the robustness of equilibrium predictions to the specification of player beliefs (Rubinstein, 1989; Monderer and Samet, 1989; Carlsson and van Damme, 1993; Kajii and Morris, 1997; Weinstein and Yildiz, 2007), but considers questions of robustness motivated by a learning foundation. This learning-based approach shares features with Dekel et al. (2004) and Esponda (2013), with a key difference that players in the present paper learn about payoffs only and not actions. Another precedent is Steiner and Stewart (2008), which characterizes the limiting equilibria of a sequence of games in which players infer payoffs from related games. I consider a related setting, where the main question is what happens when players infer payoffs using heterogeneous rules. Additionally, the focus on large data limits in Section 6 is especially related to Cripps et al. (2008), which characterizes the beliefs of Bayesian learners as the quantity of data grows large. The primary object of study in this paper is strategic behavior instead of beliefs. Finally, the depiction of agents as “statisticians” or “machine learners” relates to Gilboa and Schmeidler (2003), Gayer et al. (2007), Al-Najjar (2009), Al-Najjar and Pai (2014), Acemoglu et al. (2015), Spiegler (2016), Cherry and Salant (2017) and Olea et al. (2017), among others.

2 Example

I illustrate the main ideas below in a specialized setting. Two investors decide whether to invest in a new product. There is an underlying demand process

\[ D(t) = \beta_0 + \beta_1 t + \ldots + \beta_d t^{d^*} + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2) \]  

(1)

of unknown order \( d^* \in \mathbb{Z}_+ \) and with unknown coefficient vector \( \beta \in \mathbb{R}^{d^*+1} \). Players commonly observe demand realizations at \( n \) times sampled uniformly from an interval \( [0, T] \)

\( (D(t_1), \ldots, D(t_n)) \)

and use this to form beliefs about demand at a later time \( T > T \). Payoffs are described by the following standard coordination game

<table>
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<tr>
<th></th>
<th>Invest</th>
<th>Don’t Invest</th>
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<tr>
<td>Invest</td>
<td>( y, y )</td>
<td>( y - 1, 0 )</td>
</tr>
<tr>
<td>Don’t Invest</td>
<td>( 0, y - 1 )</td>
<td>( 0, 0 )</td>
</tr>
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</table>

where \( y = D(T) \).

Rationalizability of investment in this game depends on our model of players’ beliefs about \( y \), and also their beliefs about others’ beliefs. The standard approach to modeling these beliefs fixes a common perception of the relationship between demand realizations and the possible values of \( y \). This implies that players interpret information in the same way. When
all information is public (as it is here), investors must then share the same beliefs regardless of the number of historical observations \( n \) and also regardless of the pattern of these observations. For example, commonality of beliefs is predicted whether the historical data resembles the left or right panel in the figure below, although we may intuitively expect that coordination is harder given historical data in the right panel.

The proposed approach supposes instead that there is a set of reasonable ways to learn from the data. For example, players may agree that polynomial regression is a reasonable way to extrapolate from this data, but allow for different assumptions about the polynomial class. Formally, for each \( d \in \mathbb{Z}_+ \), let \( \hat{D}_d \) be the best \( d \)-th order polynomial fit to the observed data with \( \hat{D}_d(T) \) the corresponding prediction for demand at time \( T \).\(^9\) Instead of fixing a common prior over this set of demand processes, let us suppose only that players commonly know that the order of the polynomial is at least some \( \underline{d} \) and not more than some \( \overline{d} \); thus, they have common certainty in the set \( \{ \hat{D}_d(T) \} \) for \( \underline{d} \leq d \leq \overline{d} \). What can we predict given this assumption on beliefs alone?

In such a setting, the actual model order \( d^* \) (from (1)), the size of the model class (as determined by \( d \) and \( \overline{d} \)), and the number of common observations \( n \) are all crucial.

In fact, if \( \overline{d} = \infty \), then there is no number of observations given which we can guarantee investment (without making further assumptions on beliefs). This is because for every sequence \( D(t_0), \ldots, D(t_n) \), there is a polynomial of order \( d' \) that perfectly interpolates between the observed sequence and some \( D(T) < 0 \). In this sense, the model class is too rich, and common learning fails (see Section 5).

Suppose \( d^* \leq \underline{d} < \infty \), so that players either use correctly specified or overspecified regression models.\(^{10}\) Then, players will indeed commonly learn the true value of \( y \). As in the common prior case, given sufficiently many observations, analysis of this game can be reduced to analysis of the limiting complete information game. So long as \( y > 0 \) (so that investment is strictly rationalizable in the limiting complete information game), then investment will be

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9 Assume that players minimize least-squares error, although this choice is not critical.
10 All plausible regression models contain either exactly the right set of predictors, or at least one extraneous predictor.
rationalizable given a large number of observations (see Section 6). However, the number of observations necessary is increasing in each of \( d^* \) and \( \overline{d} \) (holding the other parameters fixed). This is because the rate at which players commonly learn is made slower both by the true underlying complexity of the model (the true order \( d^* \)), and also by the number of extraneous variables used to fit the data (Hastie et al., 2009).

Finally suppose that \( \overline{d} \leq d^* < \infty \), so that all players use correctly specified or underspecified regression models.\(^{11}\) In this case, players will generally not commonly learn the true value of \( y \). Prediction that investment is rationalizable may nevertheless make sense: the key question in this case is whether there exists an \( N \) such that for every number of observations \( n \geq N \), each under-specified model of order \( d < \overline{d} \) predicts \( y > 0 \). If so, then even without common learning, we have that investment will be rationalizable given a sufficiently large number of observations (see Section 6).

Thus, we may not want to predict investment even if \( y > 0 \) and all agents use models that will eventually recover \( y \). The strength of this prediction depends on the underlying model class and data-generating process, as well as on the number of public observations. In particular, if the set of such models is too large, or the rate at which they (jointly) recover \( y \) is too slow, then we may expect investment not to occur.\(^{12}\)

The following section develops a general framework in which players commonly observe data, and have beliefs that are consistent with a common model class. I then use this framework to provide metrics for the strength of various strategic predictions.

3 Setup and Notation

3.1 Basic Game

Consider a game with a finite set of players \( \mathcal{I} \) and a finite set of actions \( A_i \) for each player \( i \), where \( A = \times_i A_i \) is the set of action profiles. The state space is a compact subset of the set of possible payoff matrices \( \Theta \subset \mathbb{R}^{|A||\mathcal{I}|} \), endowed with the Euclidean distance metric. I will refer to \( \theta \) alternatively as the state, the payoffs, or the (complete information) game. Notice that somewhat unusually, the state \( \theta \) is the payoff matrix, and not a parameter that the payoff function depends on. To evoke more familiar notation, I often write \( u_i(a, \theta) \) for \( \theta_i(a) \); that is, the payoffs that player \( i \) receives from action profile \( a \) when the payoff matrix

\(^{11}\)All regression models miss one or more predictors.

\(^{12}\)In Figure 2, the true model order \( d^* \) is larger in the right panel (fifth-degree) than the left (linear); for fixed \( \overline{d} \) and \( \overline{d} \), and for a fixed number of realizations, the divergence in possible predictions following a typical sequence generated by the linear model is smaller than following a typical sequence generated by the higher-order polynomial. Correspondingly, we should be more cautious about predicting joint investment given the demand data on the right.
is \( \theta \). Finally, for every belief \( \mu \in \Delta(\Theta) \), the expected payoff matrix is \( E_\mu[\theta] \).

### 3.2 Description of Beliefs

Because the state space \( (\Theta, \| \cdot \|_2) \) is complete and separable, we can construct a full description of player uncertainty over \( \Theta \) as follows (Brandenburger and Dekel, 1993). Suppose first for simplicity that there are two players, and recursively define \( X_0 = \Theta \), \( X_1 = X_0 \times (\Delta(X_0)) \), \( X_2 = X_1 \times (\Delta(X_1)) \), etc., so that each \( X_k \) is the set of possible \( k \)-th order beliefs. Let \( T_0 = \prod_{n=0}^{\infty} \Delta(X_n) \). An element \( (t_1^1, t_1^2, \ldots) \in T_0 \) is a hierarchy of beliefs over \( \Theta \) (describing the player’s uncertainty over \( \Theta \), his uncertainty over his opponents’ uncertainty over \( \Theta \), and so forth), and will be referred to simply as a type.

The above approach can be generalized for \( I \) players, taking \( X_0 = \Theta \), \( X_1 = X_0 \times (\Delta(X_0)) \), \( X_2 = X_1 \times (\Delta(X_1)) \), etc., so that each \( X_k \) is the set of possible \( k \)-th order beliefs. Let \( T_i = \prod_{n=0}^{\infty} \Delta(X_n) \). An element \( (t_1^1, t_1^2, \ldots) \in T_i \) is a hierarchy of beliefs over \( \Theta \) (describing the player’s uncertainty over \( \Theta \), his uncertainty over his opponents’ uncertainty over \( \Theta \), and so forth), and will be referred to simply as a type.

The tuple \( (T_i, \kappa_i)_{i \in I} \) is known as the universal type space. Other tuples \( (T_i, \kappa_i)_{i \in I} \) with \( T_i \subseteq T_i^* \) for every \( i \), and \( \kappa_i : T_i \rightarrow \Delta(\Theta \times T_{-i}) \), represent alternative (smaller) type spaces. Since I consider only symmetric type spaces in which there exists a set \( T \) such that \( T_i = T \) for every \( i \), the set \( T \) itself will be informally referred to as the type space, with the understanding that it is identified with \( (T_i, \kappa_i)_{i \in I} \).

**Remark 1.** Types are sometimes modeled as encompassing all uncertainty in the game. In this paper, types describe only structural uncertainty over payoffs (and not strategic uncertainty over opponent actions).

### 3.3 Common \( p \)-Belief

Let \( T^* = T_1^* \times \cdots \times T_I^* \) denote the set of all type profiles, with typical element \( t = (t_1, \ldots, t_I) \). Then, \( \Omega = \Theta \times T^* \) is the set of all “states of the world.” Following Monderer and Samet (1989), for every \( E \subseteq \Omega \), let

\[
B_p(E) := \{ (\theta, t) : \kappa_i(t)(E) \geq p \text{ for every } i \},
\]

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\(^{13}\) \text{marg}_{X_{n-2}} t_i^n = t_i^{n-1}, \text{ so that } (t_1^1, t_1^2, \ldots) \text{ is a consistent stochastic process.}

\(^{14}\) For more than two players, the statement that each \( T_i = T \) should be understood as saying that each \( T_i \) is equivalent to \( T \) under an appropriate permutation of player indices.
describe the event in which every player assigns at least probability $p$ to $E \subseteq \Omega$. Common $p$-belief in the set $E$ is given by the event

$$C^p(E) := \bigcap_{k \geq 1} [\mathcal{B}^p]^k(E).$$

The special case of common 1-belief is referred to in this paper as common certainty.

I use in particular the concept of common certainty in a set of first-order beliefs, characterized in Battigalli and Siniscalchi (2003). For any $F \subseteq \Delta(\Theta)$, define

$$E_F := \{ (\theta, t) : \text{marg}_\Theta t_i \in F \text{ for every } i \},$$

(3)
to be the event in which every player’s first-order belief is in $F$. Then, $C^1(E_F)$ is the event in which it is common certainty that every player has a first-order belief in $F$. The set of types $t_i$ given which player $i$ believes that $F$ is common certainty is the projection of $C^1(E_F)$ onto $T_i^*$. Since this set is the same for all players (up to permutations on player indices), I will refer to the projection of $C^1(E_F)$ onto $T_1^*$ as “the set of types with common certainty in $F$.”

### 3.4 Solution Concepts

Fix a complete information game with payoffs $\theta$. The action profile $a$ is a Nash equilibrium in this game if for every player $i$,

$$u_i(a_i, a_{-i}, \theta) \geq u_i(a_i', a_{-i}, \theta) \quad \forall a_i' \in A_i,$$

and it is a strict Nash equilibrium if the inequality above is strict for every $a_i' \neq a_i$.

The family of sets of actions $(R_j)_{j \in \mathcal{I}}$, where every $R_j \subseteq A_j$, is closed under best reply if for every player $j$ and action $a_j \in R_j$, there is some distribution $\alpha_{-j} \in \Delta(R_{-j})$ such that

$$u_j(a_j, \alpha_{-j}, \theta) \geq u_j(a_j', \alpha_{-j}, \theta) \quad \forall a_j' \in A_j.$$

The family $(R_j)_{j \in \mathcal{I}}$ is closed under strict best reply if the inequality above holds strictly for every $a_j' \neq a_j$. An action $a_i$ is rationalizable for player $i$ if $a_i \in R_i$ for a family $(R_j)_{j \in \mathcal{I}}$ that is closed under best reply, and $a_i$ is strictly rationalizable if $a_i \in R_i$ for some family $(R_j)_{j \in \mathcal{I}}$ that is closed under strict best reply.

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15 Notice that when beliefs are allowed to be wrong (as they are here), individual perception of common certainty is the relevant object of study. That is, player $i$ can believe that a set of first-order beliefs is common certainty, even if no other player in fact has a first-order belief in this set. Conversely, even if every player indeed has a first-order belief in $F$, player $i$ may believe that no other player has a first-order belief in this set.
Now, fix an incomplete information game with type space \((T_i, \kappa_i)_{i \in I}\), so that a strategy for player \(i\) is a measurable function \(\sigma_i : T_i \to A_i\). The strategy profile \((\sigma_1, \ldots, \sigma_I)\) is a Bayesian Nash equilibrium if

\[
\sigma_i(t_i) \in \text{argmax} \int_{\Theta \times T_{-i}} u_i(a_i, \sigma_{-i}(t_{-i}), \theta) d\kappa_i(t_i)
\]

for every \(i \in I\) and \(t_i \in T_i\), so that every action \(a_i\) is a best reply to the strategy \(\sigma\) and the belief \(\kappa_i\) over player types.

I will use the following incomplete information notion of rationalizability: For every player \(i\) and type \(t_i\), set \(S^0_i[t_i] = A_i\), and define \(S^k_i[t_i]\) for \(k \geq 1\) such that \(a_i \in S^k_i[t_i]\) if and only if \(a_i \in BR_i(\text{marg}_{\Theta \times A_{-i}} \pi)\) for some \(\pi \in \Delta(\Theta \times T_{-i} \times A_{-i})\) satisfying (1) \(\text{marg}_{\Theta \times T_{-i}} \pi = \kappa_i(t_i)\) and (2) \(\pi(a_{-i} \in S^{k-1}_{j}[t_{-i}]) = 1\), where \(S^{k-1}_{-i}[t_{-i}] = \prod_{j \neq i} S^{k-1}_{j}[t_{-j}]\). We can interpret \(\pi\) to be an extension of belief \(\kappa_i(t_i)\) onto the space \(\Delta(\Theta \times T_{-i} \times A_{-i})\), with support in the set of actions that survive \(k - 1\) rounds of iterated elimination of strictly dominated strategies for types in \(T_{-i}\). For every \(i\), the actions in

\[
S^\infty_i[t_i] = \bigcap_{k=0}^\infty S^k_i[t_i]
\]

are interim correlated rationalizable for player \(i\) of type \(t_i\), or (henceforth) simply rationalizable (Dekel et al., 2007).

4 Approach

4.1 Restriction on Beliefs

The proposed approach derives a set of beliefs from two new primitives: a data-generating process, and a set of rules for how to extrapolate beliefs from realized data.

The data-generating process is formally a sequence of random variables \(Z = (Z_t)_{t \geq 1}\). Players commonly observe the realizations of the first \(n\) random variables \(Z_1, \ldots, Z_n\), which I refer to as a data set, but may interpret it in different ways. I use the convention that \(Z^n = (Z_1, \ldots, Z_n)\) is the random sequence of the first \(n\) realizations, \(z_n\) is a typical realization of \(Z^n\), and \(Z_n\) is the set of possible realizations of \((Z_1, \ldots, Z_n)\). Subscripts indicating the number of observations are dropped when they are not important.

Under the common prior assumption, players learn from data by Bayesian updating a common prior over the states and observations. I generalize this idea to learning rules, where a learning rule is any map from data sets into first-order beliefs:

\[
\mu : \bigcup_{n=1}^\infty Z_n \to \Delta(\Theta).
\]
Given data set $z_n$, every learning rule $\mu$ produces a first-order belief $\mu(z_n)$; naturally, the beliefs produced by different learning rules need not be the same. Throughout, I restrict consideration to learning rules that further satisfy the following two regularity conditions:

**Assumption 1** (Convergence). $\mu(Z^n)$ almost surely converges (in the weak topology) to some limiting belief $\mu^\infty$.

**Assumption 2** (Richness). Let $\mu^\infty$ be the limiting distribution defined above. There exists a sequence of positive numbers $\xi_n$ such that

$$\xi_n \cdot (\mu(Z^n) - \mu^\infty) \rightarrow^d \nu$$

where $\nu$ assigns strictly positive measure to every open set (in the weak topology) in a neighborhood of the zero measure.

Assumption 1 says that beliefs induced by learning rule $\mu$ almost surely converge to a limiting belief as the quantity of observations increases. Assumption 2 requires that this convergence occurs with positive probability “from any direction”. This is a weak technical condition, which loosely guarantees that the path of convergence is not too asymmetric around the limiting beliefs; in particular, players cannot over- or under-estimate any payoffs with probability 1.\textsuperscript{16,17} Both assumptions are satisfied by essentially all learning rules and data-generating processes that come up in practice.

Typically it is assumed that players interpret data using the same learning rule $\mu$, so that given identical information, players hold identical beliefs. The main departure in this paper is to allow for a set of learning rules $\mathcal{M}$. For example:

**Example A: Bayesian Updating with Different Models**

Players observe signal realizations from a set $\mathcal{Z}$. Different learning rules correspond to different models for the signal-generating distribution. Let $\mathcal{M} = \{\mu_i\}$, where each learning rule $\mu_i$ is identified with a prior distribution $\pi_i$ over $\Theta \times \mathcal{Z}^\infty$. Given data $z$, the belief induced by learning rule $\mu_i$ is the marginal of the posterior belief over $\Theta$ (updating from $\pi_i$ and $z$).

\textsuperscript{16}A more detailed explanation of Assumption 2 is as follows. Note that for each realization of data $z_n$, the expression $\xi_n \cdot (\mu(z_n) - \mu^\infty)$ is a (scaled, signed) measure over $\Theta$, describing the differences between belief $\mu(z_n)$ and the limiting belief $\mu^\infty$. This difference is the zero measure when the two beliefs are the same. Assumption 2 requires that for $n$ arbitrarily large, there is a neighborhood around the zero measure in which every open set receives strictly positive probability. This guarantees that not only does $\mu(z^n)$ converge to $\mu^\infty$ a.s., but the path of its convergence cannot exclude any particular direction of convergence with probability 1.

\textsuperscript{17}Notice that biased estimators are permitted as learning rules—for example, every learning rule may over-estimate payoffs in $\theta$ in expectation. Assumption 2 simply requires that this overestimation does not occur with probability 1.
Example B: Case-Based Learning with Different Similarity Functions

Suppose that $\mathcal{X} \subseteq \mathbb{R}$ is a set of attributes relevant to payoffs (e.g. physical covariates of a patient seeking health insurance). Players commonly observe a sequence of attribute vectors and the associated payoffs:

$$z_n = (x_1, \theta_1), \ldots, (x_n, \theta_n).$$

Attribute vectors are drawn i.i.d. from a known distribution, and payoffs are determined according to an unknown mapping $f$; that is, each $\theta_k = f(x_k)$. The attribute vector $x^*$ describing the present game is known, but the payoffs are not.

Define $\mathcal{M}$ to be a set of learning rules, where each learning rule corresponds to an approach for weighting past observations. Let $g_i : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ be a similarity function on attributes, so that $g_i(x, x')$ describes the distance between attributes $x$ and $x'$. The predicted payoff matrix at $x^*$ is the weighted average

$$\frac{1}{n} \sum_{k=1}^n \theta_k \left( e^{-\lambda g_i(x_k, x^*)} \right) / \left( \sum_{k'} e^{-\lambda g_i(x_k', x^*)} \right).$$

Each $\mu_i$ maps $z_n$ into a degenerate belief on the corresponding weighted average.

Example C: Linear Regression

As in Example 2, suppose that each observation is an attribute vector $x \in \mathbb{R}^p$ and an associated payoff $\theta$. Let the relationship between attributes and observed payoffs be given by $\theta = f(x) + \epsilon$, where $f$ is a linear function, and $\epsilon \sim \mathcal{N}(0, \epsilon^2)$ is a noise term. Payoffs depend on the value of $f(x^*)$, where the attribute vector $x^*$ is known.

Each learning rule $\mu \in \mathcal{M}$ corresponds to a different regression model, namely, some subset of attributes $I_\mu \subseteq \{1, \ldots, p\}$. Write $\hat{f}_\mu$ for the linear function of these attributes $\{x_i\}_{i \in I_\mu}$ that best fits the observed data $z_n$. Each $\mu$ maps $z_n$ into a degenerate belief on the corresponding prediction $\hat{f}_\mu(x^*)$.

Fix an arbitrary set $\mathcal{M}$ of learning rules satisfying Assumptions 1 and 2. Even if players observe a common dataset $z$, learning rules $\mu \in \mathcal{M}$ may determine different, plausible, first-order beliefs $\mu(z)$. The set of expected payoffs induced by learning rules from $\mathcal{M}$ is given by

$$\Theta(z) := \{ \mathbb{E}_{\mu(z)}[\theta] : \mu \in \mathcal{M} \}.$$

Throughout, I will consider these the possible payoffs given $z$. The main restriction imposed on beliefs is the following.

Assumption 3 (Common Inference). Let $z$ be the realization of data. Then, all agents have common certainty in the set of payoff matrices $\Theta(z)$. 

12
That is, all players assign probability 1 to payoffs in $\Theta(z)$, believe with probability 1 that all other players assign probability 1 to $\Theta(z)$, and so forth.\footnote{This is a $\Delta$-rationalizable set in the sense of Battigalli and Siniscalchi (2003), where $\Delta = \{\mu(z) : \mu \in \mathcal{M}\}$.} Notice that when $\mathcal{M}$ consists of a single Bayesian learning rule, then Common Inference reduces to the common prior assumption.

I emphasize that Common Inference is a restriction on final beliefs over $\Theta$, and does not directly build in any structural assumptions about use of learning rules. For example, we can construct beliefs consistent with Common Inference by supposing that each player $i$ uses a different learning rule $\mu_i$ to form his first-order beliefs, and these learning rules $(\mu_i)_{i \in I}$ are common knowledge. Alternatively, players may hold beliefs that are induced by a convex combination of learning rules in $\mathcal{M}$, and they may also have uncertainty over which learning rules are used by other players to form beliefs.

A key feature of Common Inference is that it permits types with common knowledge disagreement: players may know that (all know that...) they hold different first-order beliefs.\footnote{For a simple example, suppose it is common knowledge that player 1 forms first-order beliefs using $\mu_1$, and player 2 forms first-order beliefs using a different learning rule $\mu_2$.} Such types are not permitted under the common prior assumption, even when we allow for private and different information (Aumann, 1976). In this way, Common Inference represents a relaxation of the common prior assumption, where the permitted extent of disagreement is governed by a fixed set of belief updating rules.

The key question of this paper is the following: What are the restrictions on strategic behavior that are imposed by a set of learning rules $\mathcal{M}$, a data-generating process $Z$, and Common Inference?

### 4.2 Robust and Plausible Strategic Predictions

Below I propose concepts for the predictions that the analyst can make regarding equilibria and rationalizable actions, when players commonly observe a (random) dataset of size $n$ and have beliefs that obey Common Inference.

It will be useful to write $T(z)$ for the set of types satisfying Common Inference when the realized data is $z$. Say that a prediction is “robust” if it can be made for all player types from $T(z)$, and “plausible” if it can be made for some player types from $T(z)$.

**Robust Predictions.** For every action profile $a$, define $\prod z_n^{NE}(a)$ to be the probability (over possible datasets $z_n$) that play of $a$ constitutes a Bayesian Nash equilibrium whenever players
have types in $T(z_n)$;\(^{20}\) that is,

$$\bar{p}_n^{NE}(a) = \Pr\left( \{z_n : \sigma_i(t_i) = a_i \; \forall \; i, t_i \in T(z_n) \text{ is a BNE.}\} \right) \quad (4)$$

Then, $\bar{p}_n^{NE}(a)$ is a continuous measure of how likely it is that $a$ is an equilibrium when players observe $n$ realizations. Informally, the larger $\bar{p}_n^{NE}(a)$ is, the more confidence an analyst should have in predicting that $a$ is an equilibrium.

Similarly, define $\bar{p}_n^R(i, a_i)$ to be the probability (over possible datasets $z_n$) that action $a_i$ is rationalizable for player $i$ given any type in $T(z_n)$; that is,

$$\bar{p}_n^R(i, a_i) = \Pr\left( \{z_n : a_i \in S_i^Z[t_i] \; \forall \; t_i \in T(z_n)\} \right). \quad (5)$$

Again, the larger $\bar{p}_n^R(a)$ is, the more confidence an analyst should have in predicting that $a_i$ is rationalizable.

**Plausible Predictions.** In contrast to the approach taken above, consider the following less restrictive approach. Define

$$\bar{p}_n^{NE}(a) = \Pr\left( \{z_n : \exists \text{ belief-closed type space } (T_i, \kappa_i)_{i \in \mathcal{I}} \right.

\text{s.t. each } T_i \subseteq T(z_n), \text{ and } \sigma_i(t_i) = a_i \; \forall \; i, t_i \in T_i \text{ is a BNE.}\} \right) \quad (6)$$

This is the probability (over possible datasets $z_n$) that constant play of $a$ constitutes a Bayesian Nash equilibrium for some (belief-closed) type space, where all player types are from $T(z_n)$. Similarly, define

$$\bar{p}_n^R(i, a_i) = \Pr\left( \{z_n : a_i \in S_i^Z[t_i] \text{ for some } t_i \in T(z_n)\} \right). \quad (7)$$

This is the probability (over possible datasets $z_n$) that action $a_i$ is rationalizable for player $i$ of some type in $T(z_n)$.

In the special case in which agents have a common prior, these definitions have the following simple interpretation:

**Example 1.** (Common Prior.) Suppose that players share a common and correct prior over $\Theta \times \mathcal{Z}^\infty$. Write $\mu$ for the learning rule that maps any sequence of realizations $z_n = (z_1, \ldots, z_n)$ into the induced posterior belief under the common prior.

Each realization $z_n$ determines an interim game, where players all have common certainty in the induced posterior. (Notice that the set of plausible types $T(z_n)$ is a singleton for every $z_n$.) The common prior determines a distribution over data sets $z_n$, and hence a distribution

\(^{20}\)The implicit type space is $(T_i, \kappa_i)_{i \in \mathcal{I}}$ where each $T_i = T(z_n)$. Notice that the definition of $T(z_n)$ implies that this type space is belief-closed.
over possible interim games. The probabilities $\overline{p}_{n}^{NE}(a) = \overline{p}_{n}^{NE}(a)$ are equal, and they are the measure of size-$n$ datasets $z_n$ (under the common prior) with the property that action profile $a$ is an equilibrium in the corresponding interim game. Similarly, $\overline{p}_{n}^{R}(i, a_i) = \overline{p}_{n}^{R}(i, a_i)$ is the measure of size-$n$ datasets $z_n$ (under the common prior) with the property that action $a_i$ is rationalizable for player $i$ in the corresponding interim game. Note that the equivalences $\overline{p}_{n}^{NE}(a) = \overline{p}_{n}^{NE}(a)$ and $\overline{p}_{n}^{R}(i, a_i) = \overline{p}_{n}^{R}(i, a_i)$ are a consequence of $\mathcal{M}$ being a singleton. In general, these probabilities will not be the same.

4.3 Interpretation of Model

I discuss below different interpretations of the framework described above. One perspective is that the analyst knows the set of frameworks used by agents to learn about a particular economic unknown, but there is randomness in the outcome of the data. He wants to predict which actions agents might take after observing the realized data.

Under a second interpretation, the data-generating process and rules for learning from data are the “natural” measure over interim types in the absence of a common prior. When players have different priors, or do not use Bayesian updating to form beliefs, then the usual approach of defining an ex-ante distribution over interim types using the common prior is not viable. The approach outlined in Section 4.2 produces a way to quantify which interim types are more likely, and the probabilities defined in Section 4.2 then evaluate the strength of a prediction by the probability that players have types given which the prediction holds.

There is, additionally, a natural measure-theoretic notion of genericity in this framework, where a set of interim types are “generic” if this set has probability 1 under the data-generating process. This notion of genericity contrasts with the topological notion of genericity taken in many recent works (Dekel et al., 2006; Weinstein and Yildiz, 2007; Ely and Peski, 2011), where typicality means denseness in a particular topology. A set of types that has probability 1 (under the data-generating process in the present framework) can fail to be dense in the natural topologies, and likewise a dense set of types can receive probability 0 in the present framework. Finally, relative to these previous approaches, one distinguishing feature of the probabilities defined in (4-7) is that they provide continuous measures of typicality (where larger probabilities correspond to “more typical”).

5 Common Learning

Following Cripps et al. (2008), say that players commonly learn a distribution $P$ if they have asymptotic common certainty in $P$. Formally, for every probability $p \in [0, 1)$ and level of precision $\epsilon > 0$, players commonly learn $P$ if every $\epsilon$-neighborhood of $P$ (in the weak
Definition 1 (Common Learning). Players commonly learn the distribution \( P \) if

\[
\lim_{n \to \infty} \mathbb{P}(T(Z^n) \subseteq C^p(\{P\}^\epsilon)) = 1 \quad \forall \ p \in (0, 1), \ \epsilon > 0
\]

where \( \{P\}^\epsilon \) is the \( \epsilon \)-neighborhood of \( P \) (in the weak topology).

Clearly, for common learning to occur, each individual learning rule \( \mu \in \mathcal{M} \) must eventually deliver a belief arbitrarily close to \( P \); that is, \( \lim_{n \to \infty} d(\mu(Z^n), P) = 1 \) for every \( \mu \in \mathcal{M} \), where \( d \) is the Prokhorov metric on \( \Delta(\Theta) \).\(^{21}\) The stronger condition that

\[
\sup_{\mu \in \mathcal{M}} d(\mu(Z^n), P) \to 0 \ a.s.
\]

requires not only that beliefs induced by learning rules in \( \mathcal{M} \) weakly converge to \( P \), but that this convergence is uniform across \( \mathcal{M} \). Notice that this is a joint assumption on the data-generating process and the set of learning rules.

The proposition below says that (8) is the only requirement for common learning: players commonly learn \( P \) if and only if beliefs induced by learning rules in \( \mathcal{M} \) uniformly weakly converge to \( P \).

Proposition 1. Players commonly learn the true distribution \( P \) if and only if (8) holds.

When the condition in (8) is satisfied by some \( P \), I refer to \( P \) as a limiting common prior.

Definition 2 (Limiting Common Prior.). Say that \( P \in \Delta(\Theta) \) is a limiting common prior if it satisfies the condition in (8).

Common learning is a strong property; for example, it is not satisfied by the sequences of types considered in in Weinstein and Yildiz (2007), Carlsson and van Damme (1993), and Kajii and Morris (1997).\(^{22}\) The reason we see it here is because players have common certainty in the set \( \Theta(z) \) at any \( z \), which translates a restriction on first-order beliefs into a restriction on tail beliefs. As the quantity of data increases, not only does the set of plausible first-order beliefs shrink (as a direct consequence of (8)), but the set of plausible beliefs of every order shrinks uniformly across orders. Formally, the set of types \( T(Z^n) \) almost surely converges to the type with common certainty of \( P \), where convergence is in the Hausdorff metric induced by the uniform-weak metric (Chen et al., 2010) on the universal type space.

\(^{21}\)Otherwise, the type that has common certainty in the belief induced by \( \mu \) does not eventually have common \( p \)-belief in small neighborhoods of \( P \).

\(^{22}\)The analogue of \( n \to \infty \) is to take the size of the perturbation to 0.
6 Strategic Behavior: Asymptotics

Let us begin by considering the limiting strategic predictions that can be made when players have commonly observed a large number of realizations. Say that predictions are “robust to inference” if they eventually hold in all plausible interim games with probability arbitrarily close to 1.

Definition 3. Say that the equilibrium property of action profile \( a \) is robust to inference if \( \bar{p}^{NE}_n(a) \to 1 \) as \( n \to \infty \). Say that the rationalizability of action \( a_i \) for player \( i \) is robust to inference if \( \bar{p}^{R}_n(i, a_i) \to 1 \) as \( n \to \infty \).

Thus, strategic predictions are robust to inference if the analyst believes that the prediction holds with high probability given sufficient data. Conversely, if the prediction is not robust to inference, then there exists a constant \( \delta > 0 \) such that for any finite quantity of data, the probability that the prediction fails for some types consistent with Common Inference is at least \( \delta \). In this way, robustness to inference is a minimal requirement for a prediction to not require assumption that players have beliefs coordinated by an infinite quantity of data.

We can additionally ask when it is the case that a prediction holds for some player types consistent with Common Inference, so long as the number of observations is sufficiently large. I define below the analogous concept of “plausibility under inference” to capture this.

Definition 4. Say that the equilibrium property of action profile \( a \) is plausible under inference if \( \bar{p}^{NE}_n(a) \to 1 \) as \( n \to \infty \). Say that the rationalizability of action \( a_i \) for player \( i \) is plausible under inference if \( \bar{p}^{R}_n(i, a_i) \to 1 \) as \( n \to \infty \).

Section 6.1 characterizes robustness and plausibility under inference for the solution concept of Nash equilibrium, and Section 6.1 provides characterizations for rationalizability. In both cases, I begin by considering general sets of learning rules, and then turn to the special case of a limiting common prior (Assumption 2).

6.1 Equilibrium

For a given equilibrium prediction \( a \), robustness to inference is completely characterized by whether the set of plausible payoffs is eventually contained in the following set:

Definition 5. Let \( \Theta^{NE}_a \) be the set of all payoffs \( \theta \) with the property that action profile \( a \) is a Nash equilibrium when \( \theta \) is common knowledge.

The interior of \( \Theta^{NE}_a \) consists of all payoffs given which \( a \) is a strict Nash equilibrium, and its boundary consists of all payoffs given which \( a \) is a weak (and not strict) Nash equilibrium.
Lemma 1. The equilibrium property of action profile \( a \) is robust to inference if and only if

\[
\mathbb{P}(\Theta(Z^n) \subseteq \Theta_{NE}^a) \to 1 \text{ as } n \to \infty
\]  

Notice that failure of asymptotic learning (beliefs converge to an “incorrect” distribution over payoffs), and also failure of asymptotic agreement (players disagree even given infinite data), can both be consistent with robustness to inference. What is necessary and sufficient is that players are eventually certain that the expected payoffs are in \( \Theta_{NE}^a \), know that they are all certain of this, and so forth.

To better understand the condition in (9), let us consider the special case of a limiting common prior (Assumption 2).

Proposition 2. Suppose there is a limiting common prior \( P \). Then, the equilibrium property of action profile \( a \) is robust to inference if and only if \( a \) is a strict Nash equilibrium in the incomplete information game with common prior \( P \).

Thus, prediction of \( a \) is robust to inference if and only if \( a \) is a strict NE in the limiting common prior game. This result recalls Monderer and Samet (1989), which showed that strict equilibria in a complete information game are robust to approximate common certainty of payoffs; here, I consider the related exercise of weakening common certainty in a belief \( P \) to common certainty in a (shrinking) neighborhood of a belief \( P \). (Note that in contrast to Monderer and Samet (1989), players are permitted to assign probability 0 to the actual payoffs \( \theta \) arbitrarily far along the converging sequence.)

Intuitively, under the assumption of a limiting common prior, it is eventually approximate common certainty that the true game is nearby to the limiting common prior game. If action profile \( a \) is a strict equilibrium in that limiting game, then that action profile continues to be an equilibrium given common certainty in the set of nearby payoffs. Conversely, if \( a \) is only a weak equilibrium (or not an equilibrium at all), then no matter the number of observations, there is strictly positive probability (bounded below) that \( a \) is not an equilibrium. The technical condition introduced in Assumption 2 is necessary for this latter “only if” direction; for example, if players were permitted to have beliefs that consistently overestimated the payoffs to \( a \), then \( a \) would be robust to inference even if it were only a weak equilibrium in the limiting game.

The next results characterize the weaker condition of plausibility under inference. First define:

Definition 6. For each player \( i \) and action profile \( a \), let \( B_i(a_i, a_{-i}) \subseteq \Theta \) be the set of all payoff matrices given which \( a_i \) is a best response to \( a_{-i} \).
The lemma below says that prediction of \(a\) is plausible under inference so long as some payoff in each \(B(a_i, a_{-i})\) is eventually plausible.

**Lemma 2.** The equilibrium property of profile \(a\) is plausible under inference if and only if
\[
P(\Theta(\mathbb{Z}^n) \cap B_i(a_i, a_{-i}) \neq \emptyset \quad \forall i) \to 1 \quad \text{as} \quad n \to \infty
\]  

(10)

This condition is considerably weaker than the requirement that eventually \(\Theta(\mathbb{Z}^n) \cap \Theta_{NE}^a \neq \emptyset\). That is, action profile \(a\) can be a plausible equilibrium even if players agree that there it would not be played given complete information of the true payoffs. This is demonstrated in the example below:

**Example 2.** Consider the 2-player game
\[
\begin{array}{cc}
l & r \\
u & x & -x & 0,0 \\
d & 0,0 & 1,1 \\
\end{array}
\]

where \(x \in \{-1,1\}\). Players share a uniform common prior and observe signals from \(S = \{s_L, s_H\}\), which are generated according to the following information structure:

\[
\begin{array}{ccc}
s_L & s_H \\
x = -1 & p & 1-p \\
x = +1 & 1-p & p \\
\end{array}
\]

Different learning rules \(\mu_p\) correspond to Bayesian updating from different values of \(p\) in the information structure above. Set \(\mathcal{M} = \{\mu_p\}_{p \in P}\) where \(P = \left[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right]\) for some \(\epsilon > 0\).

For any sequence of realizations from \(\{s_L, s_H\}\), there is a learning rule in \(\mathcal{M}\) that assigns higher posterior belief to \(x = -1\), and a learning rule that assigns higher posterior belief to \(x = 1\). Thus, it is always consistent with Common Inference that player 1’s expected value of \(x\) is positive, but player 2’s expected value of \(x\) is negative, and these beliefs are common knowledge. So, action profile \((u, l)\) is a plausible equilibrium for all sequences of signals. But in either possible state of the world (\(x = -1\) and \(x = 1\)), the action profile \((u, l)\) is not a Nash equilibrium in the corresponding complete information game.

Notice however that in Example 2, the different learning rules yield increasingly divergent beliefs as the quantity of data gets large. Assumption of a limiting common prior rules this out; in this case, it turns out that the only surviving predictions are Nash equilibria in the limiting game.

**Proposition 3.** Suppose there is a limiting common prior \(P\). Then, the equilibrium property of action profile \(a\) is plausible under inference if and only if \(a\) is a Nash equilibrium in the incomplete information game with common prior \(P\).
Thus, under Assumption 2, the difference between the less permissive notion of \textit{robustness to inference} and the more permissive notion of \textit{plausibility under inference} grows small.

### 6.2 Rationalizability

Again, I begin by considering the more stringent condition of robustness to inference. Following the previous section, define $\Theta^R_{a_i}$ to be the set of all complete information games in which some action $a_i$ is rationalizable.

**Definition 7.** For each player $i$ and action $a_i \in A_i$, let $\Theta^R_{a_i}$ be the set of all payoffs given which action $a_i$ is rationalizable for player $i$.

The earlier results for equilibrium (Lemma 1 and Proposition 2) suggest the following parallel statements for rationalizability: (1) robustness to inference is characterized by whether the set of plausible expected payoffs is eventually contained in $\Theta^R_{a_i}$; and (2) if there is a limiting common prior, then robustness to inference is characterized by whether $a_i$ is strictly rationalizable in the limiting game. Neither statement turns out to hold.

Let us begin by discussing (1). Notice that although $a_i$ is rationalizable at every payoff in $\Theta^R_{a_i}$, the chain of best responses rationalizing action $a_i$ can vary across the set. The simple example below shows that even if $a_i$ is rationalizable in each of two different games, it can fail to be rationalizable when the player has uncertainty over which of the two payoffs will be realized. This is closely related to the well-known fact that the set of rationalizable actions is not convex.

**Example 3.** Consider a two-player game. Player 1 assigns equal probability to the following two games:

\[
\begin{array}{ccc}
  a_3 & a_4 & \ \ \\
  a_1 & 1, 1 & -1, 0 \\
  a_2 & 0, 1 & 1, 0 \\
\end{array}
\quad
\begin{array}{ccc}
  a_3 & a_4 & \ \ \\
  a_1 & -1, 0 & 1, 1 \\
  a_2 & 1, 0 & 0, 1 \\
\end{array}
\]

In the first game, action $a_3$ is uniquely rationalizable action for player 2, and in the second, action $a_4$ is uniquely rationalizable. So player 1 believes with probability 1/2 that player 2 will choose $a_3$, and with probability 1/2 that player 2 will choose $a_4$. Thus player 1’s expected payoffs are given by

\[
\begin{array}{ccc}
  a_3 & a_4 & \ \ \\
  a_1 & 0, 1/2 & 0, 1/2 \\
  a_2 & 1/2, 1/2 & 1/2, 1/2 \\
\end{array}
\]

and $a_2$ is his uniquely rationalizable action. This follows even though $a_1$ is a rationalizable action in both games.
Thus, common certainty in $\Theta^{R}_{a_i}$ does not imply rationalizability of $a_i$. In fact, even common certainty in an arbitrarily small open set within $\Theta^{R}_{a_i}$ does not guarantee rationalizability of $a_i$ (see an example demonstrating this in Appendix B.5). The basic intuition is that this set of payoffs may span two sets of payoff functions with different families of rationalizable actions. Actions that are rationalizable when players perceive different, arbitrarily close, payoffs, need not be rationalizable given any common perception of payoffs.

One way to remedy this is to require eventual common certainty in an open set of payoffs, across which the chain of best responses rationalizing action $a_i$ remains constant. This condition turns out to be unnecessarily restrictive, and the example below demonstrates that it is not necessary for robustness to inference.

**Example 4.** Consider the following complete information game

<table>
<thead>
<tr>
<th></th>
<th>$a_3$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0,0</td>
<td>0,0</td>
</tr>
</tbody>
</table>

and notice that $a_1$ is rationalizable. Moreover, since $a_1$ is strictly dominant at all nearby payoffs (in Euclidean distance), $a_1$ remains rationalizable given common certainty of a small enough neighborhood of these payoffs. But consider the following perturbations:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_3$</td>
<td>$a_4$</td>
<td>$a_3$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>1, $-\epsilon$</td>
<td>1,0</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0, $-\epsilon$</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Action $a_1$ remains rationalizable in both games, but $a_1$ is not a best reply to $a_3$ in the game on the left, and $a_1$ is not a best reply to $a_4$ in the game on the right. Thus, there is no chain of best responses rationalizing $a_1$ that holds on any (arbitrarily small) neighborhood of the original payoffs. Relatedly, action $a_1$ turns out not to be strictly rationalizable in the original game.

Lemma 3 organizes the above observations. It says that rationalizability of an action is robust to inference if players eventually have common certainty in some set of payoffs, across which $a_i$ can be rationalized using the same best response chain, and only if players eventually have common certainty in $\Theta^{R}_{a_i}$.

**Lemma 3.** The rationalizability of action $a_i$ for player $i$ is robust to inference if

$$\mathbb{P}(\Theta(Z^n) \subseteq V) \to 1 \text{ as } n \to \infty$$

(11)

\footnote{A nice two-player example in the concurrent work of Chen and Takahashi (2017) shows this as well.}
for some set $V$, where $a_i$ can be rationalized using the same chain of best-responses for all payoffs in $V$, and only if
\[
P(\Theta(Z^n) \subseteq \Theta_{R_i}^{R}) \to 1 \text{ as } n \to \infty.
\] (12)

Again, to better understand these conditions, let us consider the case in which there is a limiting common prior (Assumption 2). In this case, the sufficient condition above reduces to strict rationalizability of $a_i$ in the limiting game, and the necessary condition reduces to weak-strict rationalizability, a new property that I now introduce:

Recall that strict rationalizability can be defined as the limit of a process of iterative elimination of actions that are never a strict best reply. It is well known that this procedure is sensitive to the manner of elimination. Consider specifically all the orders of elimination in which at most one action is eliminated at a time. Formally, define $W_i^1 := A_i$ for every player $i$. Then, for each $k \geq 2$, recursively remove (at most) one action in $W_i^k$ that is not a strict best reply to any opponent strategy $\alpha_{-i} \in \Delta(W_{-i}^{k-1})$. Let
\[
W_i^\infty = \bigcap_{k \geq 1} W_i^k
\]
be the set of player $i$ actions that survive every round of elimination, and define $W_i^\infty$ to be the intersection of all sets $W_i^\infty$ that can be constructed in this way.

**Definition 8.** Say that an action $a_i$ is weakly strict-rationalizable if $a_i \in W_i^\infty$.

Returning to Example 4, we see that there are two patterns of one-at-a-time elimination. One possibility is
\[
\begin{array}{ccc}
a_3 & a_4 & a_3 & a_4 \\
a_1 & 1,0 & 1,0 & a_1 & 1,0 \\
a_2 & 0,0 & 0,0 & a_2
\end{array}
\]
in which action $a_2$ is eliminated for player 1 and action $a_4$ is eliminated for player 2, so that actions $a_1$ and $a_3$ remain. Another possibility is
\[
\begin{array}{ccc}
a_3 & a_4 & a_3 & a_4 \\
a_1 & 1,0 & 1,0 & a_1 & 1,0 \\
a_2 & 0,0 & 0,0 & a_2
\end{array}
\]
in which action $a_2$ is eliminated for player 1 and action $a_3$ is eliminated for player 2, so that actions $a_1$ and $a_4$ remain. The action $a_1$ survives both procedures; hence, it is weakly strict-rationalizable.

Weak-strict rationalizability turns out to be the right condition because it characterizes the interior of $U_{i,a_i}^R$; that is, action $a_i$ is weakly-strict rationalizable in every game in the interior of $U_{i,a_i}^R$ (see Lemma 5 in the appendix). From this, and Lemma 3, it follows that:
Proposition 4. Suppose there is a limiting common prior $P$. Then, rationalizability of action $a_i$ for player $i$ is robust to inference if $a_i$ is strictly rationalizable in the incomplete information game with common prior $P$, and only if $a_i$ is weakly-strict rationalizable in the incomplete information game with common prior $P$.

Observe that refinement is obtained despite the negative results of Weinstein and Yildiz (2007). One way to understand this is to observe that (with probability 1) any sequence of types $t^n$ from $T(Z^n)$ converges in the uniform-weak topology (Chen et al., 2010), while the main result in Weinstein and Yildiz (2007) relies on types that converge only in the (coarser) product topology; see Section 10 for an extended discussion. Loosely speaking, the tail beliefs of types in Weinstein and Yildiz (2007) are permitted to put high probability on payoff functions that receive low probability at all lower orders. In the present paper, players have common certainty in a (shrinking) set of payoffs, so higher order beliefs are required to have similar supports to the lower order beliefs.

The sufficiency direction of this result is shown directly from lower hemi-continuity of strict rationalizability in the uniform-weak topology (Chen et al., 2010), although the requirement that sequences of types converge uniformly over a given set requires more than lower hemi-continuity alone. The stronger requirement imposed by robustness to inference turns out to be equivalent to the concept of robustly rationalizable (Morris et al., 2012) under the assumption of a limiting prior\textsuperscript{24}. The necessity direction, and in particular the role of weak strict-rationalizability, requires new arguments. Discussion of other related work is provided in Section 10.

Turning now to plausibility of inference, for every distribution $\nu \in \Delta(\Theta \times A_i)$, define $BR_i(\nu) := \arg\max_{a_i \in A_i} \mathbb{E}_\nu u(a_i, a_{-i}, \theta)$ to be player $i$’s set of best replies given belief $\nu$.

Lemma 4. Rationalizability of action $a_i$ for player $i$ is plausible under inference if and only if

$$\mathbb{P}\left( \exists \text{ family } (R_i)_{i \in I} \text{ s.t. } \forall i, a_i \in BR_i(\nu_i) \right. \left. \text{ for some } \nu_i \in \Delta(\Theta(Z^n) \times R_{-i}) \right) \to 1 \text{ as } n \to \infty$$ (13)

Again, let us restrict to a limiting common prior (Assumption 2).

Proposition 5. Suppose there is a limiting common prior. Then, rationalizability of action $a_i$ for player $i$ is robust to inference if and only if $a_i$ is rationalizable in the incomplete information game with common prior $P$.

\textsuperscript{24}The present setting differs from this prior work in that $\Theta$ may not be finite, and the notion of robustness is across all types with approximate common certainty in neighborhoods of the true parameter. These differences are not crucial for the characterization of asymptotic strategic behavior, but they do matter for the rate results in the subsequent section.
Thus with large quantities of data, we again see that the difference between the less permissive notion of *robustness to inference* and the more permissive notion of *plausibility under inference* is small. These concepts diverge substantially in the next section, where we consider small quantities of data.

7 Strategic Behavior: Finite-Sample

Propositions 2 and 4 tell us that when there is a limiting common prior, and players observe a large quantity of data, then it is largely without loss to analyze the limiting common prior game instead. Specifically, predictions eventually hold for all plausible types only if they hold “strictly” in the limiting common prior game, and they eventually hold for some plausible types only if they hold in the limiting common prior game.

These conclusions no longer follow when the number of common observations is small. Below, I focus on this setting of small $n$, and quantify the probability with which the corresponding equilibrium (and rationalizable action) sets differ from the limiting game. Here I emphasize the utility of $\overline{p}_{a}^{NE}$ (and the related measures defined in Section 4) as continuous metrics; for example, we can ask how large $\overline{p}_{a}^{NE}$ is for a fixed action profile $a$ and number of observations $n$. Informally, we should have more confidence in predictions that hold with high probability for small $n$, than predictions that only hold when players observe a large number of data points.

Throughout this section, I impose two additional simplifying assumptions. First, observations are i.i.d. from a finite set:

**Assumption 4.** $Z_1, \ldots, Z_n \sim_{\text{i.i.d.}} Q$ and take values in a finite set $\mathcal{Z}$.

Second, there is a limiting common prior that is degenerate on a payoff matrix $\theta^*$; thus, the limiting game (as we take the number of observations to infinity) is a complete information game.

**Assumption 5.** There is a limiting common prior that is degenerate on payoffs $\theta^*$.

Assumption 4 is important for the statement of the bounds, while Assumption 5 can be relaxed to assumption of any limiting common prior $P$ (as in Assumption 2), at the expense of additional notation.

7.1 Robust Predictions

The previous sections have shown that whether a solution holds strictly determines asymptotic robustness; in this section, I use cardinal metrics for “how” strict a solution is.
First, say that an action profile $a$ is a $\delta$-strict Nash equilibrium if $u_i(a_i, a_{-i}, \theta^*) > u_i(a_i', a_{-i}, \theta^*) + \delta$ for every player $i$ and action $a_i' \neq a_i$.\(^{25}\) The parameter

$$\delta_a := \sup \{ \delta : a \text{ is a } \delta\text{-strict NE in the game } \theta^* \}$$

quantifies the degree of strictness, and we will be interested in the case in which $\delta_a > 0$. The larger the parameter $\delta_a$ is, the larger the neighborhood of $\theta^*$ on which $a$ remains an equilibrium.

The analogous parameter for rationalizability is this: Say that an action $a_i$ is $\delta$-strictly rationalizable for player $i$ if there exists a family of sets $(R_j)_{j \in \mathcal{I}}$ with $a_i \in R_i$, such that for every player $j$ and action $a_j \in R_j$, there is some distribution $\alpha_{-j} \in \Delta(R_{-j})$ such that

$$u_j(a_j, \alpha_{-j}, \theta^*) > u_j(a_j', \alpha_{-j}, \theta^*) + \delta \quad \forall a_j' \neq a_j. \quad (14)$$

Again, we can similarly define

$$\delta_{i,a_i} := \sup \{ \delta : a_i \text{ is } \delta\text{-strictly rationalizable in the game } \theta^* \} .$$

When positive, this parameter determines the maximal neighborhood of $\theta^*$ over which $a_i$ is rationalizable (using the same chain of best responses).

Proposition 6 uses these parameters to lower bound $p^\text{NE}_n(a)$ and $p^R_n(i, a_i)$ for all quantities of data $n$.

**Proposition 6.** If $a$ is a strict Nash equilibrium in the complete information game with payoffs $\theta^*$, and $a_i$ is rationalizable for player $i$, then

$$p^\text{NE}_n(a) \geq 1 - \frac{1}{\delta_a} \mathbb{E} \left( d_H(\Theta(Z^n), \theta^*) \right) \quad \forall \ n \geq 1 \quad (15)$$

and

$$p^R_n(i, a_i) \geq 1 - \frac{1}{\delta_{i,a_i}} \mathbb{E} \left( d_H(\Theta(Z^n), \theta^*) \right) \quad \forall \ n \geq 1 \quad (16)$$

where $d_H$ is the Hausdorff metric induced by the sup-norm on $\Theta$.\(^{26}\)

Thus, if $a$ is a Nash equilibrium in the limiting complete information game, and players have observed $n$ realizations, the probability that $a$ is a Bayesian Nash equilibrium is at least the probability in (15) above. Similarly, if $a_i$ is rationalizable in the limiting complete

---

\(^{25}\)This definition reverses the more familiar concept of $\epsilon$-equilibrium, which was introduced to formalize a notion of approximate Nash equilibria (violating the equilibrium conditions by no more than $\epsilon$). I use $\delta$-strict equilibrium to provide a cardinal measure for the strictness of a Nash equilibrium (satisfying the conditions with $\delta$ to spare).

\(^{26}\)Let $(X,d)$ be a metric space. For any set $A \subset X$ and element $x \in X$, their Hausdorff distance (induced by $d$) is $d_H(X,x) = \sup_{x' \in X} d(x',x)$.  

25
information game, and players have observed \( n \) realizations, then the probability that \( a_i \) is rationalizable is at least the probability in (16) above.

Two observations are in order. First, the bounds in Proposition 6 are increasing in the strictness of the parameters \( \delta_a \) and \( \delta_{i,a,i} \); thus, the “more strict” a prediction in the limiting game, the fewer observations are necessary for the prediction to hold for any \( n \).

The bounds are also decreasing in \( E d_H(\Theta(Z^n), \theta^*) \). To interpret this, first rewrite \( d_H(\Theta(Z^n), \theta^*) = \sup_{\theta' \in \Theta(Z^n)} \| \theta', \theta^* \|_{\infty} \) as the distance from the true payoffs \( \theta^* \) to the farthest plausible payoffs in \( \Theta(Z^n) \). This distance naturally depends on the realization of the data set \( Z^n \). The expression \( E d_H(\Theta(Z^n), \theta^*) \) is the expected distance, taking into account the randomness over the data. When players commonly learn (see Proposition 1), then the expression \( E d_H(\Theta(Z^n), \theta^*) \) converges to 0 as \( n \to \infty \). The bounds in Proposition 6 thus tell us that the quicker players commonly learn, the fewer observations are necessary for limiting predictions to carry over to small-data settings.

Finally, Proposition 6 is used to provide quantitative bounds below in an example setting.

**Example 5.** Consider the coordination game from Section 2:

\[
\begin{array}{ccc}
\text{a} & \text{b} \\
\text{a} & y, y & y - 1, 0 \\
\text{b} & 0, y - 1 & 0, 0 \\
\end{array}
\]

where \( y \in \mathbb{R} \) is unknown. Players observe public signals \( z_t = y + \epsilon_t \), with standard normal error terms \( \epsilon_t \) that are i.i.d. across time.

The set of possible models in \( \mathcal{M} \) are identified with different priors \( y \sim \mathcal{N}(x, 1) \), where \( x \) is in the bounded interval \([-\eta, \eta]\). Write \( \mu_x \) for the learning rule that maps the observed data to the posterior belief associated with prior \( \mathcal{N}(x, 1) \). Notice that players commonly learn the true payoffs.

Suppose that the true value of \( y > 0 \), so that action profile \((a, a)\) is an equilibrium given complete information. We know from Section 6 that \((a, a)\) will also be an equilibrium if players observe sufficient data. The corollary below provides a lower bound on the probability that \((a, a)\) is an equilibrium when players have seen \( n \) observations.

**Corollary 1.** For each \( n \geq 1 \) and player \( i = 1, 2 \),

\[
P_{n}^{NE}(a, a) \geq 1 - \frac{1}{|y|} \left( \sqrt{\frac{2}{\pi n}} \frac{y + \eta}{n + 1} \right)
\]

Thus, for example, for the choice of parameters \( y = 2, n = 500, \) and \( \eta = 10 \), the action profile \((a, a)\) is an equilibrium with probability at least 0.97. In contrast, for the same choice of \( y \) and \( \eta \), if players observe only 15 data points, then action profile \((a, a)\) can only be guaranteed
to be an equilibrium with probability 0.50. The bound in Corollary 1 is decreasing in \( \eta \) (the size of the model class), and increasing in \( n \) (the number of observations) and \( y \) (the strictness of the equilibrim).

7.2 Plausible Predictions

The above approach can be applied to answer the question of whether a prediction which holds in the limit also holds away from the limit. We can alternatively consider predictions that don’t hold in the limit, and consider whether we should also rule out these predictions when players have observed only a few data points.

Formally, for each action profile \( a \), let \( Z_a \) be the set of all data sets given which \( a \) is a plausible equilibrium (see Lemma 10). Notice that the probability \( \tilde{p}_n^{NE}(a) \) is exactly the probability that the realized data (of size \( n \)) is in the set \( Z_a \).

Each data set \( z_n \) can be associated with an empirical measure \( \hat{Q}(z_n) \) over the signal set \( Z \). Define

\[
Q^*_a = \arg\min_{\hat{Q} \in \{\hat{Q}(z) : z \in Z_a\}} D_{KL}(\hat{Q} || Q)
\]

where \( D_{KL} \) is the Kullback-Leibler distance. The distribution \( Q^*_a \) is the empirical measure (associated with a data set in \( Z^n_a \)) that is closest in Kullback-Leibler distance to the actual signal-generating distribution \( Q \).

I will use a similar set of definitions for rationalizability. First, for each player \( i \) and action \( a_i \), define \( Z_{a_i}^R \) to be all datasets given which action \( a_i \) is plausibly rationalizable action for player \( i \) (see Lemma 13). Then let

\[
Q^*_{i,a_i} = \arg\min_{\hat{Q} \in \{\hat{Q}(z) : z \in Z_{a_i}^R\}} D_{KL}(\hat{Q} || Q)
\]

be the empirical measure (associated with a data set in \( Z_{a_i}^R \)) that is closest in Kullback-Leibler distance to the actual signal-generating distribution \( Q \).

Application of Sanov’s theorem directly gives the following corollary from Lemmas 10 and 13.

**Corollary 2.** If \( a \) is not a Nash equilibrium in the complete information game with payoffs \( \theta^* \), and \( a_i \) is not rationalizable for player \( i \), then

\[
\tilde{p}_n^{NE}(a) \leq (n + 1)^{|Z|} 2^{-nD_{KL}(Q^*_a || Q)} \quad \forall \ n \geq 1
\]

---

\( ^{27} \)The Kullback-Leibler distance from distribution \( \hat{Q} \) to \( Q \) is

\[
D_{KL}(\hat{Q} || Q) = \sum_{z \in Z} \hat{Q}(z) \log \frac{Q(z)}{\hat{Q}(z)}
\]
and

\[ p_n^R(i, a_i) \leq (n + 1)^{2-nD_{KL}(Q^*_{i,a_i}||Q)} \quad \forall \ n \geq 1 \]

These bounds are illustrated below in an example setting:

**Example 6.** Consider the game:

<table>
<thead>
<tr>
<th></th>
<th>Sell</th>
<th>Not Sell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy</td>
<td>(y - 1/2, 1/2)</td>
<td>0, (y)</td>
</tr>
<tr>
<td>Not Buy</td>
<td>0, (y)</td>
<td>0, (y)</td>
</tr>
</tbody>
</table>

where \(y\) takes values in \(\{0, 1\}\). Notice that (Buy, Sell) is not a Nash equilibrium given complete information of either possible value of \(y\).

Players observe signals from \(\{s_0, s_1\}\), where the signal-generating structure is given by

<table>
<thead>
<tr>
<th>(s_0)</th>
<th>(s_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = 0)</td>
<td>(q, 1 - q)</td>
</tr>
<tr>
<td>(y = 1)</td>
<td>(1 - q, q)</td>
</tr>
</tbody>
</table>

but \(q\) is unknown. The set of possible models in \(\mathcal{M}\) are identified with different priors over \(\{0, 1\}\) and different values for \(q\) in the information structure above. Write \(\mu_{\pi,q}\) for the learning rule that maps the observed data to the posterior belief associated with prior \((1 - \pi, \pi)\) and information structure \(q\), and define

\[ \mathcal{M} = \left\{ \mu_{\pi,q} : \pi \in \left[ \frac{1}{3}, \frac{2}{3} \right], q \in \left[ \frac{2}{3}, \frac{3}{4} \right] \right\}. \]

Notice that players commonly learn the true payoffs.

For concreteness, let us further suppose that the true value of \(q\) is \(3/4\), and the true value of \(y\) is 1. We know from Section 6 that \(a\) will fail to be plausibly rationalizable given that players have observed sufficient data. Nevertheless, action \(a\) may be plausibly rationalizable if players have observed a small number of data points, and the corollary below quantifies this.

**Corollary 3.** For each \(n \geq 1\) and player \(i = 1, 2\),

\[ p_n^{NE}(Buy, Sell) \leq (n + 1)^2 2^{-0.14n}. \]

Thus, if players have observed (for example) \(n = 100\) realizations, then the probability that \((Buy, Sell)\) is plausible equilibrium is upper bounded by 0.62, while if \(n = 100\), the probability that \((Buy, Sell)\) is a plausible equilibrium is approximately 0.
The data-generating process has so far been considered an exogenous primitive, but in practice public data is often reported by an external actor. I provide an illustration below of how the choice of what data to reveal can influence strategic behaviors within the proposed framework.

The example focuses on a special case in which there is an unknown relationship between a set of measurable characteristics and a payoff-relevant outcome. Write $\mathcal{X} = [-c, c]^p$ for the set of $p$-dimensional covariate vectors and suppose that there is a binary payoff-relevant outcome. The relationship between characteristics and outcomes is given by the unknown function:

$$f(x) = \begin{cases} 1 & \text{if } x_k \in [-c^*, c^*] \text{ for every } k \leq p^* \\ 0 & \text{otherwise} \end{cases}$$

where $c^* < c$. Notice that only the first $p^*$ characteristics matter for the outcome.

Suppose that payoffs depend on $y = f(x^*)$, the outcome associated with a particular covariate vector $x^*$. I consider the following payoffs, slightly modified from Section 2:

<table>
<thead>
<tr>
<th>Invest</th>
<th>Not Invest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invest</td>
<td>$y, 0$</td>
</tr>
<tr>
<td>Not Invest</td>
<td>$0, \frac{1}{2}$</td>
</tr>
<tr>
<td>Not Invest</td>
<td>$\frac{1}{2}, 0$</td>
</tr>
<tr>
<td>Invest</td>
<td>$1, \frac{1}{2}$</td>
</tr>
<tr>
<td>Not Invest</td>
<td>$\frac{1}{2}, \frac{1}{2}$</td>
</tr>
</tbody>
</table>

The public data regarding $y$ is determined as follows. First, covariate vectors $x$ are drawn uniformly at random from $\mathcal{X}$, and an external agency observes the pairs $(x, f(x))$. The agency then controls which characteristics are released to the public; specifically, it chooses how many coordinates of the observed covariate vectors to report. For simplicity, I assume that the agency is obligated to report each of the first $p^*$ characteristics, but can in addition (truthfully) report additional characteristics. Let the agency’s choice be any integer $p \in \{p^*, \ldots, p\}$. Players thus observe

$$z_n = \left((x_1^{1:p}, f(x_1)), \ldots, (x_n^{1:p}, f(x_n))\right),$$

where $x_1^{1:p}$ is the truncation of vector $x$ to its first $p$ entries.

It is common knowledge that the true model $f$ belongs to the set $F$ of rectangular classification rules\footnote{This set includes every function $\hat{f} : [-c, c]^p \rightarrow \{0, 1\}$ that can be written as $\hat{f}(x) = \begin{cases} 1 & \text{if } x \in [c_1, \bar{c}_1] \times \cdots \times [c_p, \bar{c}_p] \\ 0 & \text{otherwise} \end{cases}$ for constants $c_1, \bar{c}_1, \ldots, c_p, \bar{c}_p \in [-c, c]$.} but players may have different priors over this set. Let each learning rule in $\mathcal{M}$ be associated with a prior over $F$, and let the induced belief over outcomes at $x^*$
be the posterior probability corresponding to Bayesian updating from that prior.\textsuperscript{29} Suppose that \( y = f(\mathbf{x}^*) = 1 \) so that ‘Invest’ is a rationalizable action in the complete information game. Then:

**Proposition 7.** For every quantity of data \( n \geq 1 \), and for both players \( i \in \{1, 2\} \):

(a) the probability \( p^R_n(i, \text{Invest}) \) is monotonically decreasing in the number of reported characteristics \( p \).

(b) \( p^R_n(i, \text{Invest}) \to 0 \) as the number of reported characteristics \( p \to \infty \).

Thus, if the agency wants to minimize the probability that investment is rationalizable given \( n \) observations, it should report as many characteristics as possible (\( p = \bar{p} \)). Moreover, if we allow the agency to report arbitrarily many characteristics \( p \) (this requires \( \bar{p} = \infty \)), then \( p^R_n(i, \text{Invest}) \) can be made arbitrarily small for any fixed number of observations. The essential feature of this example is that players do not know which or how many characteristics \( f \) depends on. Thus, the more characteristics are reported, the greater the number of models that are “consistent” with the data, and as a result, the greater the ambiguity in how to interpret the data.

## 9 Extensions

The following section provides brief comment on various modeling choices made in the main framework.

**Misspecification.** The main results hold under a weakening of Assumption 2, which I define below:

**Definition 9** (Approximate Limiting Common Prior.). For any \( \epsilon \geq 0 \), say that the class of learning rules \( \mathcal{M} \) has a \((1 - \epsilon)\)-limiting common prior \( P \) if

\[
\limsup_{n \to \infty} \sup_{\mu \in \mathcal{M}} d(\mu(Z^n), P) \leq \epsilon \text{ a.s.}
\]

where \( d \) is the Prokhorov metric on \( \Delta(\Theta) \).

\textsuperscript{29}Formally, the set of states of the world is \( \Omega = F \times \mathcal{Z}^\infty \), so that a state consists of a function \( f \) and an infinite sequence of observations from \( \mathcal{Z} \). Let \( \mathcal{M} \) be indexed by the set of all prior beliefs over \( F \), with the property that the induced distribution over parameters in \([c, \bar{c}]\)\textsuperscript{27} is absolutely continuous with respect to the Lebesgue measure.

Conditional on a model \( \hat{f} \in F \), a stochastic process \( \psi_\hat{f} \) generates an infinite sequence of i.i.d. observations \( (x, y) \), where \( x \) is drawn uniformly at random from \( \mathcal{X} \), and deterministically \( y = f(x) \). Let \( \{\mathcal{H}_n\}|_{n=1}^\infty \) denote the filtration induced on \( \Omega \) by datasets \( \mathbf{z}_n \) of size \( n \). Then, every \( \mathbf{z}_n \) and prior belief \( \mu \) on \( F \) generates a posterior belief over outcomes \( f(\mathbf{x}^*) \).
According to this definition, the class of learning rules $\mathcal{M}$ has a $(1 - \epsilon)$-limiting common prior $P$ if the set of induced first order beliefs converges almost surely to an $\epsilon$-neighborhood of $P$. Notice that Assumption 2 is nested as the $\epsilon = 0$ case. Propositions 2 and 4 can be weakened to the following result. (In reading this, recall that if $\mathcal{M}$ has a $(1 - \epsilon)$-limiting common prior $P$, then it also has a $(1 - \epsilon')$-limiting common prior $P$ for every $\epsilon' > \epsilon$.)

**Proposition 8.** (a) Suppose there is a $(1 - \delta_a)$-limiting common prior $P$. Then, the equilibrium property of $a$ is robust to inference if and only if $a$ is a strict equilibrium in the game with common prior $P$.

(b) Suppose there is a $(1 - \delta^R_{a_i})$-limiting common prior $P$. Then, the rationalizability of action $a_i$ is robust to inference if $a_i$ is strictly rationalizable in the game with common prior $P$.

That is, the main results from Section 6 hold even if players have heterogeneous and incorrect beliefs even in the limit, so long as their limit beliefs are constrained within a $\delta^N_{NE}$ neighborhood (respectively, $\delta^R_{a_i}$-neighborhood) of the degenerate belief on $\theta^*$. The proof is straightforward and omitted.

**Private Data.** This paper studies players who observe a common dataset, but interpret it in different ways. When players observe private data, common learning may not occur even if $\mathcal{M}$ consists of a single learning rule (Cripps et al., 2008). Thus, Proposition 1 need not extend. Moreover, Carlsson and van Damme (1993) and Kajii and Morris (1997) (among others) have shown that strict Nash equilibria are not robust to higher-order uncertainty about private opponent information. Thus, Propositions 2 and 4 also will not hold without additional restrictions on beliefs.

In the simplest extension, however, we may suppose that players observe different datasets $(z^i)_{i \in I}$, independently drawn from the same distribution, but have (incorrect) degenerate beliefs that all opponents have seen the same data that they have. Then, Propositions 2 and 4 extend, and the bounds in Proposition 6 can be revised as follows (where $I$ is the number of players).

**Proposition 9.** Suppose there is a limiting common prior $P$. If $a$ is a strict Nash equilibrium in the game with common prior $P$, and $a_i$ is rationalizable for player $i$, then

$$p^N_{a_i}(a) \geq \left(1 - \frac{1}{\delta_a} \mathbb{E} \left( \sup_{\mu \in \mathcal{M}} d(\mu(Z^n), P) \right) \right)^{I} \quad \forall n \geq 1$$

and

$$p^R(i, a_i) \geq \left(1 - \frac{1}{\delta_{a_i}} \mathbb{E} \left( \sup_{\mu \in \mathcal{M}} d(\mu(Z^n), P) \right) \right)^{I} \quad \forall n \geq 1$$

where $d$ is the Prokhorov metric on $\Delta(\Theta)$.
Approximate Common Inference. Finally, suppose that instead of having common certainty in $\Theta(z)$ (as we have assumed in the main text), players have common $p$-belief in $\Theta(z)$. In this case, the rate results in Section 7 require modification, but all asymptotic results stated in Section 6 extend for $p$ sufficiently large.

10 Related Literature

This paper builds on a literature that studies the robustness of strategic predictions to the specification of player beliefs. Early work compared the strategic behaviors of types whose beliefs were close up to order $k$ for large $k$ (Rubinstein, 1989; Mertens and Zamir, 1985; Brandenburger and Dekel, 1993). Several authors have demonstrated that this notion of nearby (which corresponds to the product topology on the universal type space) leads to surprising and counterintuitive conclusions, in particular, even strict equilibria and strictly rationalizable actions are fragile to perturbations (Rubinstein, 1989; Weinstein and Yildiz, 2007).

Dekel et al. (2006), Chen et al. (2010), and Chen et al. (2017) developed and characterized finer metric topologies on types under which the desired continuity properties hold. In particular, the uniform-weak topology proposed in Chen et al. (2010) considers two types to be close if they have similar first-order beliefs, attach similar probabilities to other players having similar first-order beliefs, and so forth.

Under the assumption of a limiting common prior, all types consistent with Common Inference converge in the uniform-weak topology. Thus, the property of robustness to inference, considered in Section 6, can be interpreted as requiring robustness of strategic predictions across a subset of perturbations in the uniform-weak topology. In more detail, we can consider the following reduced version of the framework: Let $\Theta^n$ be a sequence of sets of payoff matrices, and define $T^n_i$ to be the set of player $i$ types with common certainty in $\Theta^n$. Say that the prediction that an action $a_i$ is rationalizable is robust to inference if it holds for every $t_i \in T^n_i$ for $n$ sufficiently large. This condition coincides with the concept of robust rationalizability proposed in Morris et al. (2012), and subsequently characterized in the concurrent work of Takahashi (2017) and Chen and Takahashi (2017). (In particular, this paper’s Proposition 4 is related to results in this concurrent work.) Similarly, we can say that prediction of a rationalizable action is plausible under inference if it holds for some player $i$ type in $T^n_i$ for $n$ sufficiently large. Analogous definitions exist for equilibrium.

When rates are of interest, as they are in Section 7, then the reduction above is not sufficient. In the full model, the sequence $\Theta^n$ is a random sequence, and the distribution over payoff sets $\Theta^n$ is important for determining the probability with which $a_i$ will be rationalizable for all types $t_i \in T^n_i$. Details of the primitives of the data-generating process and
set of learning rules are important for this speed of convergence.

Finally, the robustness exercise considered in this paper is related to Carlsson and van Damme (1993) and Kajii and Morris (1997), with a few key differences. At a technical level, the prior work uses contagion arguments that do not apply in my setting, and also imposes a common prior (which I do not). Collectively, these differences lead to rather different robustness results. Conceptually, the goals of the present paper are different as well, as the focus here is not on equilibrium selection (choosing one equilibrium from a set of many), but rather on providing (continuous) metrics for the strength of a given prediction.

11 Conclusion

I examine the strategic implications when multiple agents form beliefs given common certainty of a diverse model class. A set of “plausible” hierarchies of beliefs are defined from a common dataset and a set of rules for extrapolating from the data. The proposed approach is substantially more permissive than the common prior assumption, but restrictive enough still to make predictions. As the quantity of data converges to infinity, beliefs and behavior can be approximated by a limit complete information game. For small quantities of data, the appropriateness of such a reduction depends on the complexity of the problem of learning payoffs and the strictness of limit solutions.

\footnote{For example, the construction of beliefs used in Weinstein and Yildiz (2007) to show failure of robustness (Proposition 2) relies on construction of tail beliefs that place positive probability on an opponent having a first-order belief that implies a dominant action. A similar device is employed in Kajii and Morris (1997) to show that robust equilibria need not exist (see the negative example in Section 3.1). These tail beliefs are not permitted under my approach. When the quantity of data is taken to be sufficiently large, it is common certainty (with high probability) that all players have first-order beliefs close to the true distribution.}
Appendix

A Supplementary Material to Section 5

A.1 Proof of Proposition 1

Preliminaries. Let \( T_i^k = \Delta(X_{k-1}) = \Delta(\Theta \times T_{k-i}^{k-1}) \) denote the set of \( k \)-th order beliefs for player \( i \).\(^{31}\) Let \( \rho_0^i \) be the Euclidean norm on \( \Theta \), and recursively for \( k \geq 1 \), define \( \rho^k_i \) to be the Prokhorov distance\(^{32}\) on \( \Delta \left( \Theta \times T_{k-i}^{k-1} \right) \) induced by the metric \( \max\{\rho_0^i, \rho_{k-1}^i\} \) on \( \Theta \times T_{k-i}^{k-1} \). The uniform-weak metric (proposed in Chen et al. (2010)) is

\[
\rho_{UW}^i(t_i, t'_i) = \sup_{k \geq 1} \rho^k_i(t_i, t'_i).
\]

As in the main text, since only symmetric type spaces are considered, player subscripts are dropped throughout on distances, type sets, and types.

A few final notational conventions: I use \( E_\delta \) to mean the \( \delta \)-neighborhood of \( E \) (where the metric should be clear from context). Throughout, \( d \) is always used to mean the Prokhorov metric on \( \Delta(\Theta) \) induced by the Euclidean metric on \( \Theta \). Additionally, when \( z \) is an infinite sequence, I use \( z_{1:n} \) to mean the truncation of \( z \) to its first \( n \) coordinates.

Uniform convergence implies common learning (only if): The arguments in this section build on those used in Chen et al. (2010) to show equivalence between convergence in the uniform-weak topology and common learning.

Fix any dataset \( z \). We can decompose the set of types \( T(z) \) into the Cartesian product \( \prod_{k=1}^{\infty} H_z^k \), where \( H_z^1 = \Delta_z \) and for each \( k > 1 \), \( H_z^k \) is recursively defined

\[
H_z^k = \left\{ t^k \in T^k : (\text{marg}_{T^{k-1}} t^k) (H_z^{k-1}) = 1 \text{ and } \text{marg}_\Theta t^k \in H_z^1 \right\}.
\] (18)

This is the set of \( k \)-th order beliefs of types in \( T(z) \). Define

\[
\Delta_z := \{ \mu(z) : \mu \in \mathcal{M} \}
\]

to be the set of first-order beliefs induced by learning rules in \( \mathcal{M} \), and define

\[
\delta^* := \sup_{P \in \Delta_z} d(P', P)
\]
to be the largest distance between \( P \) and any belief in \( \Delta_z \) (where \( d \) is the Prokhorov metric on \( \Delta(\Theta) \)). I show below that for every dataset \( z \), the distance between any type \( t \in T(z) \) and the type \( t_P \) with common certainty of \( P \) is no more than \( \delta^* \).

Claim 1. \( \rho^k(t, t_P) \leq \delta^* \) for every \( k \geq 1 \) and \( t \in T(z) \),

\(^{31}\)Working only with types in the universal type space, it is possible to identify each \( X_k \) with its first and last coordinates, since all intermediate information is redundant.

\(^{32}\)Recall that the Levy-Prokhorov distance \( \rho \) between measures on metric space \((X, d)\) is defined

\[
\rho(\mu, \mu') = \inf \{ \delta > 0 : \mu(E) \leq \mu'(E^\delta) + \delta \text{ for each measurable } E \subseteq X \}
\]

for all \( \mu, \mu' \in \Delta(X) \), where \( E^\delta = \{ x \in X : \inf_{x' \in E} \rho(x, x') < \delta \} \).
Proof. Fix any $t \in T(z)$. By construction of $T(z)$, the first-order belief of type $t$ is in the set $\Delta_z$. So it is immediate that

$$\rho^1(t, tp) \leq \sup_{P' \in \Delta_z} d(P', P) = \delta^*.$$  \hfill (19)

Now suppose $H_z^k \subseteq \{t_p^k\}^\delta^*$. Consider any measurable set $E \subseteq T^k$. If $t_p^k \in E$, then $t_p^{k+1}(E) = 1$ by definition of $t_p$. Also

$$t^{k+1} \left(E^{\delta^*}\right) \geq t^{k+1} \left(\{t_p^k\}^\delta^*\right) \geq t^{k+1} \left(H_z^k\right) = 1,$$

using (18) in the final equality and the inductive assumption in the inequality preceding it. So

$$t^{k+1}(E) \leq t^{k+1} \left(E^{\delta^*}\right) + \delta^*.$$  \hfill (20)

If $t_p^k \notin E$, then $t_p^{k+1}(E) = 0$ (again by definition of $t_p$), so (20) follows trivially. Thus $t_p^{k+1}(E) \leq t^{k+1}(E^{\delta^*}) + \delta^* $ for every measurable $E \subseteq T^k$. Using this and (19), $\rho^{k+1}(t, tp) \leq \delta^*$ as desired. \hfill $\square$

The claim immediately implies that $\rho^{UW}(t, tp) \leq \delta^*$ for all types $t \in T(z)$.

Now consider an arbitrary $\epsilon > 0$ and $p \in [0, 1)$ and choose $\delta \leq \min(\epsilon, 1 - p)$. I will show that $\{P\}^\epsilon$ is common $p$-belief for all types in $\{t_p\}^\delta$. Fix an arbitrary $t \in \{t_p\}^\delta$. Trivially,

$$\operatorname{marg}_\Theta t \in \{P\}^\delta \subseteq \{P\}^\epsilon.$$  

Moreover,

$$t^1(\{P\}^\epsilon) \geq t^1(\{P\}^\delta) \geq 1 - \delta \geq p$$

where the second inequality follows from $\rho^1(t, tp) \leq \delta$. Thus, every type in $\{t_p\}^\delta$ assigns at least probability $p$ to the set of first-order beliefs $\{P\}^\epsilon$, or $\{t_p\}^\delta \subseteq \mathcal{B}^p(\{P\}^\epsilon)$. Now suppose that $\{t_p\}^\delta \subseteq [\mathcal{B}^p]^k(\{P\}^\epsilon)$. By a similar argument,

$$t^{k+1}(\mathcal{B}^p)^k(\{P\}^\epsilon) \geq t^{k+1}(\{t_p^k\}^\delta) \geq 1 - \delta \geq p,$$

So also $\{t_p\}^\delta \subseteq [\mathcal{B}^p]^k(\{P\}^\epsilon)$. Thus, $\{t_p\}^\delta \subseteq [\mathcal{B}^p]^k(\{P\}^\epsilon)$ for every order $k$, and it follows that $\{t_p\}^\delta \subseteq C^p(\{P\}^\epsilon)$ as desired.

Under condition (8), the set

$$\left\{z : \lim_{n \to \infty} \sup_{P' \in \Delta_{z_{1:n}}} d(P', P) = 0\right\}$$

has measure 1. Thus, for every $\epsilon > 0$ and $p \in [0, 1)$, there is a measure-1 set of sequences $z$ such that $\{P\}^\epsilon$ is common $p$-belief for all types in $T(z_{1:n})$ when $n$ is sufficiently large.

Common learning implies uniform convergence (if): Suppose condition (8) is not satisfied; then, there exists some $\epsilon > 0$ such that

$$Z^* := \left\{z : \lim_{n \to \infty} \sup_{\mu \in \mathcal{M}} d(\mu(z_{1:n}), P) > \epsilon\right\}$$

has positive measure. For every $z \in Z^*$, we can construct a sequence of first-order beliefs $\nu^n \to P$, where each $\nu^n = \mu(z_{1:n})$ for some $\mu \in \mathcal{M}$. Let $t^n$ be the sequence of types with common certainty in $\nu^n$. Then, there exists an $N > 0$ and $p \in (0, 1)$ such that the set $\{P\}^\epsilon$ is not common $p$-belief for any type $t^n$ with $n > N$. So common learning fails with probability bounded away from zero.

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33By definition, $t_p^1(\{P\}) = 1$, and since $\rho^1(t, tp) \leq \delta$, necessarily $t_p^1(\{P\}) \leq \rho^1(\{P\}) + \delta$, which implies that $\rho^1(\{P\}) \geq 1 - \delta$.
B Supplementary Material to Section 6

Throughout, I use $\sigma_a$ to denote the strategy profile constant on $a$: that is, $\sigma_i(t_i) = a_i$ for every player $i$ and type $t_i \in T_i$. For simplicity of exposition, the dependence of $\sigma_a$ on its type space is not made explicit.

B.1 Proof of Lemma 1

Consider the event in which the set of plausible payoffs satisfies $\Theta(z_n) \subseteq \Theta_a^{\text{NE}}$. Since every player’s first-order belief has support in $\Theta_a^{\text{NE}}$, each $a_i$ is a best reply to $a_{-i}$ (given player $i$’s beliefs). Thus, $\Theta(z_n) \subseteq \Theta_a^{\text{NE}}$ is a sufficient condition for the strategy profile $\sigma_a$ to be a Bayesian Nash equilibrium, and it follows that the assumption in (9) directly implies that $p_{n}^{\text{NE}}(a) \to 1$.

For the other direction, suppose that for all $n$ sufficiently large, the measure of datasets

$$\{z_n : \Theta(z_n) \not\subseteq \Theta_a^{\text{NE}}\}$$

is at least some constant $\delta > 0$ (uniform over $n$). Conditioning on the event that $\Theta(z_n) \not\subseteq \Theta_a^{\text{NE}}$, there exists a payoff matrix $\theta \in \Theta(z_n) \setminus \Theta_a^{\text{NE}}$. In this game $\theta$, $u_i(a_i, a_{-i}, \theta) < u_i(a'_i, a_{-i}, \theta)$ for some player $i$ and action $a'_i \in A_i$. So, action $a_i$ is not a best reply against $a_{-i}$ when player $i$ has common certainty in $\theta$. But since $\theta \in \Theta(z_n)$, the type with common certainty in $\theta$ is contained in the set $T(z_n)$. Thus, if $\Theta(z_n) \not\subseteq \Theta_a^{\text{NE}}$, the strategy profile $\sigma_a$ fails to be a BNE for some plausible vector of types from $T(z_n)$. Since the event $\Theta(z_n) \not\subseteq \Theta_a^{\text{NE}}$ occurs with positive probability (uniformly bounded away from zero across all $n$), we have that $p_{n}^{\text{NE}}(a) \to 1$, concluding the proof.

B.2 Proof of Proposition 2

Suppose $a$ is a strict Nash equilibrium with common prior $P$. Then, the expected payoff matrix $E_P(\theta)$ is in the interior of $\Theta_a^{\text{NE}}$. Moreover, since the space of payoffs $\Theta$ is bounded, our assumption that $\sup_{\mu, M} d(\mu(Z^n), P) \to 0$ a.s. directly implies

$$\sup_{\mu, M} \|E_\mu(z_n)[\theta] - E_P[\theta]\|_\infty \to 0 \text{ a.s.}$$

Thus, the set $\Theta(z_n) = \{E_\mu(z_n)[\theta] : \mu \in M\}$ converges almost surely to $\{E_P(\theta)\}$ in the Hausdorff metric induced by $\| \cdot \|_\infty$. The set $\Theta(z_n)$ is almost surely contained in $\Theta_a^{\text{NE}}$ and the first part of the corollary follows from Lemma 1.

Now suppose that $a$ is not a strict NE in the game with common prior $P$, so that $E_P(\theta) \not\in \text{Int}(\Theta_a^{\text{NE}})$. By Assumption 2 (Richness), for every $n$ sufficiently large, there is a positive measure of realizations $z_n$ given which $E_P(\theta)$ is not an extremal point in $\Theta(z_n)$. Combining this with the arguments above, the probability that $\Theta(z_n)$ contains some payoff matrix $\theta \in \Theta_a^{\text{NE}}$ is bounded away from 0. Again apply Lemma 1 and we are done.

B.3 Proof of Lemma 2

Consider the event in which the set of plausible payoffs $\Theta(z_n)$ has nonempty intersection with each $B_i(a_i, a_{-i})$. Let each player $i$ have common certainty in a payoff matrix $\theta \in B_i(a_i, a_{-i})$; then, each $a_i$ is a best response to $a_{-i}$. The probability that $\Theta(z_n) \cap B_i(a_i, a_{-i})$ for all $i$ is thus a lower bound on $p_{a}^{\text{NE}}$, and the condition in (10) directly implies $p_{a}^{\text{NE}} \to 1$.

Conversely, suppose that $\Theta(z_n) \cap B_i(a_i, a_{-i}) = \emptyset$ for some player $i$. Then, there is no first-order belief with support in $\Theta(z_n)$ given which player $i$ perceives $a_i$ to be a best reply to $a_{-i}$. Thus, failure of (10) implies that $p_{a}^{\text{NE}} \to 1$. 
B.4 Proof of Proposition 2

Suppose there is a limiting common prior \( P \). Then, the set of plausible payoffs \( \Theta(z_n) \rightarrow \{ E_P(\theta) \} \). Under condition (10), necessarily \( E_P(\theta) \in B(a_i, a_{-i}) \) for every \( i \), implying that \( a \) is a Nash equilibrium in the limiting game with common prior \( P \). Conversely, suppose that \( a \) is not a Nash equilibrium in the limiting game with common prior \( P \). Then, there is some player \( i \) for whom \( E_P(\theta) \notin B(a_i, a_{-i}) \). This further implies that
\[
\{ E_P(\theta) \}^\epsilon \in B(a_i, a_{-i}) \text{ for some } \epsilon > 0
\]
where \( \{ E_P(\theta) \}^\epsilon \) is the \( \epsilon \)-neighborhood of the payoff matrix \( E_P(\theta) \). Since \( \Theta(Z^n) \subseteq \{ E_P(\theta) \}^\epsilon \) with probability arbitrarily close to 1 for \( n \) sufficiently large, it follows that condition (10) does not hold.

B.5 Example

Consider the following four player game. Players 1 and 2 choose between actions in \( \{ a, b \} \), and player 3 chooses between matrices from \( \{ l, r \} \). The expected payoffs for players 1-3 are given by:

\[
\begin{array}{ccc}
a & b & a & b \\
a & 1,1,0 & 0,0,0 & 0,0,0 & 0,0,0 \\
b & 0,0,0 & 0,0,0 & 0,0,0 & 1,1,0 \\
(l) & (r)
\end{array}
\]

A fourth player chooses between \{Match, Mismatch\}, where his action is not relevant to the other players' payoffs. Player 4 receives a payoff of 1 from Match if players 1 and 2 choose the same action, and he receives 0 otherwise. He receives a payoff of 1 from Mismatch if players 1 and 2 choose different actions, and 0 otherwise.

In every game with payoffs sufficiently close\(^{35}\) to those above, Match is rationalizable for player 4, because one (or both of) actions \( a \) and \( b \) is simultaneously rationalizable for player 1 and 2.\(^{36}\) Nevertheless, I will show that Match fails to be rationalizable for a sequence of types with common certainty in increasingly small neighborhoods of the payoffs described above. The basic idea is that player 4 can believe that \( a \) is uniquely rationalizable for player 1, while \( b \) is uniquely rationalizable for player 2. To see this, consider the following two perturbed versions of the payoff matrix above:

\[
\begin{array}{ccc}
a & b & a & b \\
a & 1,1,0 & 0,0,0 & 0,0,0 & 0,0,0 \\
b & 0,0,0 & -\epsilon,0,0 & b & 0,0,-\epsilon & 1,1,\epsilon \\
(l) & (r)
\end{array}
\]

\[
\begin{array}{ccc}
a & b & a & b \\
a & 1,1,-\epsilon & 0,0,0 & 0,0,0 & 0,0,0 \\
b & 0,0,-\epsilon & 0,0,-\epsilon & b & 0,0,0 & 1,1,0 \\
(l) & (r)
\end{array}
\]

\(^{34}\)This convergence occurs in the Hausdorff metric induced by \( \| \cdot \|_\infty \).

\(^{35}\)e.g. in the Euclidean metric.

\(^{36}\)If neither \( l \) nor \( r \) are strictly dominated for Player 1, then all actions are rationalizable for player 1-3, and if either \( l \) or \( r \) is strictly dominated for player 1, then one of the following will be a rationalizable family for player 1-3: \( \{ l \} \times \{ a \} \times \{ a \} \), \( \{ l \} \times \{ a, b \} \times \{ a, b \} \), \( \{ r \} \times \{ b \} \times \{ b \} \), or \( \{ r \} \times \{ a, b \} \times \{ a, b \} \).
Let $\epsilon > 0$. If player 1 has common certainty in the payoffs in (22), then $a$ is his uniquely rationalizable action: $l$ strictly dominates $r$ for player 3, and in the surviving game, $a$ strictly dominates $b$ for player 1. By a similar argument, if player 2 has common certainty in the payoffs in (23), then $b$ is his uniquely rationalizable action. These statements hold for $\epsilon$ arbitrarily small. Construct a sequence of types $t_\epsilon$ for player 4, where each type $t_\epsilon$ has common certainty that player 1 has common certainty in (22) and player 2 has common certainty in (23). Then, player 4 of type $t_\epsilon$ has only one rationalizable action, Mismatch. Take $\epsilon \to 0$ and the desired conclusion obtains.

### B.6 Proof of Lemma 3

To show sufficiency of the condition in (11), suppose that for all payoffs in the set $V$, action $a_i$ is rationalizable using the same chain of best responses. Then there exists a family $(R_j)_{j \in I}$ closed under strict best reply for all games in $V$, where action $a_i \in R_i$. Clearly these relations are preserved when all players have first-order beliefs with support in $V$. Thus, for any sequence of types $(t_1, \ldots, t_I)$ with common certainty in $V$, there exists a family of sets $R_j \subseteq A_j$ where every action $a_j \in R_j$ is a best reply to a distribution $\pi \in \Delta(\Theta \times T_{-j} \times A_{-j})$ satisfying $\text{marg}_{\Theta \times T_{-j}} \pi = g(t_j)$ and $\pi(a_{-j} \in R_{-j}[t_{-j}]) = 1$, and also $a_i \in R_i$. The desired conclusion that action $a_i$ is rationalizable for player $i$ then follows from the result below:

**Proposition 10** (Dekel et al. (2007)). Fix any type profile $(t_j)_{j \in I}$. Consider any family of sets $R_j \subseteq A_j$ such that every action $a_j \in R_j$ is a best reply to a distribution $\pi \in \Delta(\Theta \times T_{-j} \times A_{-j})$ that satisfies $\text{marg}_{\Theta \times T_{-j}} \pi = \kappa_j(t_j)$ and $\pi(a_{-j} \in R_{-j}[t_{-j}]) = 1$. Then, $R_j \subseteq S^\pi_j[t_j]$ for every player $j$.

The condition in (11) thus guarantees that $p^R_n(a_i) \to 1$, so rationalizability of action $a_i$ is robust to inference.

To show necessity of the condition in (12), consider the event in which data $z_n$ is realized such that $\Theta(z_n) \not\subseteq \Theta^R_{a_i}$. Then, there is a plausible payoff matrix $\theta \not\in \Theta^R_{a_i}$, and a plausible belief with common certainty in $\theta$. Clearly action $a_i$ is not rationalizable for player $i$ with this belief, so we are done.

### B.7 Proof of Proposition 4

Suppose $a_i$ is strictly rationalizable in the game $\theta$. Then, there exists a family $(R_j)_{j \in I}$ of sets from $A_j$ with the property that $a_i \in A_i$, and for every player $j$, each action $a_j \in R_j$ is a strict best response to some distribution over $R_{-j}$.

Players will eventually have common certainty of the payoffs in $\Theta^R_{a_i}$ only if the limiting game $\theta$ is in the interior of $\Theta^R_{a_i}$. I show below that $\theta$ is in the interior of $\Theta^R_{a_i}$ only if $a_i$ is weakly strictly-rationalizable in the limiting game $\theta$. This follows from the stronger result below, which says that weak strict rationalizability characterizes the interior of $\Theta^R_{a_i}$.

**Lemma 5.** $\theta \in \text{Int}(\Theta^R_{a_i})$ if and only if $a_i$ is weakly strict-rationalizable in the complete information game with payoffs $\theta$.

**Proof.** If: Suppose the game $\theta$ is not in the interior of $\Theta^R_{a_i}$. There must then exist a sequence $\theta^n \to \theta$ (converging in the sup-norm), where for large $n$, the game $\theta^n$ is also not in the interior of $\Theta^R_{a_i}$. Thus in each late game $\theta^n$, there is an order of elimination of strictly dominated strategies that removes $a_i$. Moreover, since action sets are finite, there is a finite number of possible such orders of elimination. This implies existence of a subsequence along which the same order of iterated elimination of strategies removes $a_i$. At the limiting payoffs $\theta$, action $a_i$ must fail to
survive elimination of weakly dominated strategies along this order, and is therefore not weakly strict-rationalizable.

*Only if:* Suppose $a_i$ is not weakly strict-rationalizable. Then, there exists a sequence of sets \( \left( W^k_j \right)_{k \geq 1} \) for every player $j$ satisfying the recursive description in Section 6.2, such that $a_i \notin W^k_i$ for some $K < \infty$. To show that $\theta$ is not in the interior of $\Theta^{R\alpha}_\alpha$, I construct a sequence of payoff functions $\theta^n$ with $\theta^n \to \theta$ such that $a_i$ is not rationalizable in any late game along this sequence.

For every $n \geq 1$, construct the payoff function $n^n$ according to the following procedure. First, for every player $j$, let $\theta^n,0 = \theta$. Then, for every $l \geq 1$, define $\theta^n,l$ such that

$$u_j(a_j, a_{-j}, \theta^n,l) = \begin{cases} u_j(a_j, a_{-j}, \theta^n,l-1) + \epsilon/n & \forall a_j \in W^l_j, a_{-j} \in A_{-j} \\ u_j(a_j, a_{-j}, \theta^n,l-1) & \forall a_j \notin W^l_j, a_{-j} \in A_{-j} \end{cases}$$

That is, we iteratively increase the payoffs of the surviving strategies at each round of elimination (according to \( W^k_j \)) by $\epsilon/n$. Finally, let $\theta^n = \theta^n,K$.

I claim that $a_i$ is not rationalizable in any complete information game $\theta^n$, when $n$ is sufficiently large. In game $\theta^n$, let $S^{k,n}_j$ be the set of player $j$ actions that survive $k$ rounds of iterated elimination of strictly dominated strategies. I will show that in games $\theta^n$ where $n$ is sufficiently large, the sets $S^{k,n}_j = W^k_j$ for all $k$ and every player $j$.

Proceed by induction. Trivially, $S^{0,n}_j = W^0_j = A_j$ for every $j$ and $n$. Suppose $S^{k,n}_j = W^k_j$ for every player $j$ and round $k \leq L$ when $n$ is sufficiently large. Now consider any action $a_j \in S^{L,n}_j$.

Suppose $a_j$ is a strict best response to some strategy $\alpha_{-j} \in \Delta(W^L_j)$; then clearly, $a_j \in W^{L+1}_j$. The claim below implies that also $a_j \in S^{L+1,n}_j$ for $n$ sufficiently large.

**Claim 2.** Suppose $a_j$ is a strict best response to some strategy $\alpha_{-j} \in \Delta(A_{-j})$ and define

$$\gamma = \frac{1}{2} \left| u_j(a_j, \alpha_{-j}, \theta) - \max_{a'_{-j} \neq a_j} u_j(a'_j, \alpha_{-j}, \theta) \right| > 0$$

Set $N > \frac{\epsilon K}{\gamma}$. Then, action $a_j$ is a strict best reply to $\alpha_{-j}$ in every game $\theta^n$ when $n > N$.

**Proof.** Define $a^*_j = \arg\max_{a'_{-j} \neq a_j} u(a'_j, \alpha_{-j}, \theta^n)$. Then,

$$u_j(a_j, \alpha_{-j}, \theta^n) - u_j(a^*_j, \alpha_{-j}, \theta^n) = u_j(a_j, \alpha_{-j}, \theta^n) - u_j(a_j, \alpha_{-j}, \theta) + u_j(a_j, \alpha_{-j}, \theta) - u_j(a^*_j, \alpha_{-j}, \theta) + u_j(a^*_j, \alpha_{-j}, \theta) - u_j(a^*_j, \alpha_{-j}, \theta^n) \geq -(\epsilon K)/n + 2\gamma - (\epsilon K)/n \geq 2\gamma - (2\epsilon K)/N > 0$$

using in the penultimate inequality that $n > N$, and in the final inequality that $N > \frac{\epsilon K}{\gamma}$. \(\square\)

Thus we have $W^{L+1}_j = S^{L+1,n}_j$ for large $n$ in this case.

Suppose that $a_j$ is only a weak best response to some strategy $\alpha_{-j} \in \Delta(W^L_j)$. In this case, there are two possibilities: (1) If $a_j \in W^{L+1}_j$, then by construction of $\theta^n$, action $a_j$ is rendered a strict best response to $\alpha_{-j}$ under $\theta^n$; thus, $a_j \in S^{L+1,n}_j$ for all $n$ sufficiently large. (2) Otherwise, if $a_j \notin W^{L+1}_j$, then there exists some alternative action $a'_j$ satisfying $u_j(a'_j, \alpha_{-j}, \theta) = u_j(a_j, \alpha_{-j}, \theta)$
and also \( a'_j \in W^{L+1}_j \). By construction of payoffs \( \theta^n \), the action \( a'_j \) yields a strictly higher payoff than \( a_j \) against \( \alpha_{-j} \) in all games \( \theta^n \). Repeating this argument for any \( \alpha_{-j} \in \Delta(W^L_j) \) to which \( a_j \) is a weak best reply, we have that \( a_j \notin S^{L+1,n}_j \) for \( n \) sufficiently large (using Claim 2). Finally, if \( a_j \) is not a weak best response to any strategy \( \alpha_{-j} \in \Delta(W^L_j) \), then both \( a_j \notin W^{L+1}_j \) and also \( a_j \notin S^{L+1,n}_j \) for \( n \) sufficiently large.

Therefore \( S^{k,n}_j = W^k_j \) for every \( k \) and \( n \) sufficiently large. Since \( a_j \notin W^K_j \), also \( a_j \notin S^{T,n}_j \) for \( n \) sufficiently large, as desired. Finally, by construction \( \theta^n \to \theta \), so \( \theta \notin \text{Int}(\Theta^R_{a_i}) \), as desired. \( \square \)

### B.8 Proof of Lemma 4

Consider the event in which there is a family \((R_i)_{i \in I}\) such that for each player \( i \) and action \( a_i \in R_i \), \( a_i \in BR_i(\nu_i) \) for some belief \( \nu_i \in \Delta(\Theta(z_n) \times R_{-i}) \). Suppose players have common certainty that each player \( i \)'s first-order belief is \( \text{marg}_\Theta \nu_i \). Then, action \( a_i \) is rationalizable for player \( i \), so the condition in (13) implies \( \mathbb{P}^R_n(i, a_i) \to 0 \) as desired.

Conversely, suppose that \( a_i \) is rationalizable for player \( i \) of some type \( t_i \in T(z_n) \). By definition of rationalizability, there exists a family of sets \( R_j \subseteq A_j \) such that every action \( a_j \in R_j \) is a best reply to a distribution \( \pi \in \Delta(\Theta \times T_{-j} \times A_{-j}) \) that satisfies \( \text{marg}_{\Theta \times T_{-j} \times A_{-j}} \pi = \kappa(t_j) \), and \( \pi(a_{-j} \in R_{-j}[t_{-j}]) = 1 \). Moreover, since \( t_i \in T(z_n) \) by assumption, we have that \( \text{marg}_{T_{-i}} \pi \) assigns probability 1 to the event that each \( t_j \in T(z_n) \). Now let each \( \nu_i = \text{marg}_{\Theta \times A_{-j}} t_i \). Since the types in \( T(z_n) \) have common certainty in \( \Theta(z_n) \), we have that \( \nu_i \in \Delta(\Theta(z_n) \times R_{-i}) \) as desired. Thus, the condition in (13) is necessary for rationalizability of \( a_i \) to be plausible under inference.

### B.9 Proof of Proposition 5

This follows directly from Lemma 4, noting that expected payoffs are continuous in first-order beliefs (with respect to the Prokhorov metric) on a compact set.

### C Supplementary Material to Section 7

#### C.1 Proof of Proposition 6

(a) To simplify notation, set \( \delta := \delta^{NE}_{a} \). Since \( a \) is by assumption a strict NE, \( \delta > 0 \). Applying Lemma 1, if \( \Theta(z_n) \subseteq \Theta^n_{a} \), then the strategy profile \( \sigma_a \) (as defined in Appendix B) is a Bayesian Nash equilibrium. Write \( Q^n \) for the product measure (associated with \( Q \)) over \( n \)-length sequences in \( \mathcal{Z}_n \). Then,

\[
p^n_{a} \geq Q^n \left( \left\{ z_n : \Theta(z_n) \subseteq \Theta^n_{a} \right\} \right)
\]

\[
= Q^n \left( \left\{ z_n : \sup_{\mu \in M} \mathbb{E}_{\mu(z_n)}[|\theta - \theta^*|_\infty] \leq \delta \right\} \right)
\]

\[
= 1 - Q^n \left( \left\{ z_n : \sup_{\mu \in M} \mathbb{E}_{\mu(z_n)}[|\theta - \theta^*|_\infty] > \delta \right\} \right)
\]

\[
\geq 1 - \frac{1}{\delta} \mathbb{E} \left( \sup_{\mu \in M} \mathbb{E}_{\mu(Z^n)}[|\theta - \theta^*|_\infty] \right) = 1 - \frac{1}{\delta} \mathbb{E} (d_H(\Theta(Z^n, \theta^*)))
\]

using Markov’s inequality in the final line.
(b) To simplify notation, set \( \delta := \delta^{R}. \) Since \( a_{i} \) is strictly rationalizable for player \( i, \) \( \delta > 0. \) Applying Lemma 3, if players have common certainty of the set \( \{ \theta : \| \theta - \theta^{*} \|_{\infty} \leq \delta \}, \) then the strategy \( a_{i} \) is rationalizable for player \( i. \) This allows us to construct the lower bound

\[
p^{R}(a_{i}) \geq Q^{n}\left( \left\{ z_{n} : \sup_{\mu \in \mathcal{M}} \| E_{\mu}(z_{n})[\theta] - \theta^{*} \|_{\infty} \leq \delta \right\} \right)
\]

\[
\geq 1 - \frac{1}{\delta} E\left( \sup_{\mu \in \mathcal{M}} \| E_{\mu}(z_{n})[\theta] - \theta^{*} \|_{\infty} \right) = 1 - \frac{1}{\delta} E\left( d_{H}(\Theta(Z^{n}), \theta^{*}) \right)
\]
as in part (a).

### C.2 Proof of Corollary 1

First observe that \( \delta_{(a,a)} = y, \) since the action profile \( (a, a) \) is a \( y \)-strict NE in the complete information game with payoffs \( \theta^{*}. \) It remains to determine \( E\left( d_{H}(\Theta(Z^{n}), \theta^{*}) \right). \) Write \( Z_{n} \) for the (random) empirical mean of \( n \) signal realizations. Then:

\[
E\left( d_{H}(\Theta(Z^{n}), \theta^{*}) \right) = E\left[ \max_{x \in [-\eta, \eta]} \left( |y - \frac{x + nZ_{n}}{n + 1}| \right) \right]
\]

\[
\leq E\left( |y - \frac{nZ_{n}}{n + 1}| + \max_{x \in [-\eta, \eta]} \left| \frac{x}{n + 1} \right| \right)
\]

\[
= E\left( |y - \frac{nZ_{n}}{n + 1}| \right) + \eta/(n + 1)
\]

\[
\leq E\left( |y - Z_{n}| \right) + E\left( \frac{Z_{n}}{n + 1} \right) + \eta/(n + 1)
\]

\[
= \frac{2}{n\pi} + \frac{y + \eta}{n + 1}
\]

using in the final line the expected absolute deviation of the empirical mean of \( n \) observations from \( \mathcal{N}(y, 1). \)

### C.3 Proof of Corollary 3

Below, identify \( \mathcal{M} \) with \([1/3, 2/3] \times [2/3, 3/4], \) the set of priors and information structures. Fix an arbitrary pair \( (\pi, q) \in \mathcal{M}, \) and a data sequence \( z_{n} \) consisting of \( m \) realizations of \( y = 1 \) and \( n - m \) realizations of \( y = 0. \) The expected value of \( y, \) updating from \( (\pi, q) \) and \( z_{n}, \) is

\[
y_{\pi,q}(z_{n}) := E(y|z_{n}) = \frac{1}{1 + \frac{1 - \pi}{\pi} \left( \frac{1 - q}{q} \right)^{2m-n}}.
\]

(24)

Notice that \( y_{\pi,q}(z_{n}) \) is monotonically increasing in \( \pi. \) When \( z_{n} \) consists of majority realizations of \( y = 1, \) then \( y_{\pi,q}(z_{n}) \) is also monotonically increasing in \( q; \) otherwise, it is monotonically decreasing in \( q. \) Therefore, for all data sets \( z_{n} \) with majority realizations of \( s_{1}, \)

\[
y_{1/3,2/3}(z_{n}) \leq y_{\pi,q}(z_{n}) \leq y_{2/3,3/4}(z_{n}) \quad \forall (\pi, q) \in \mathcal{M}
\]

and for all data sets \( z_{n} \) with majority realizations of \( s_{0}, \)

\[
y_{1/3,3/4}(z_{n}) \leq y_{\pi,q}(z_{n}) \leq y_{2/3,2/3}(z_{n}) \quad \forall (\pi, q) \in \mathcal{M}.
\]

(24)
The action Buy is a best reply to Sell if the expected value of \( y \) exceeds 1/2, and the action Sell is a best reply to Buy if the expected value of \( y \) does not exceed 1/2. Thus, the action profile \((Buy, Sell)\) is a plausible equilibrium for all data sets \( z_n \) that either satisfy

\[
\hat{y}_{1/3,2/3}(z_n) \leq 1/2 \leq \hat{y}_{2/3,3/4}(z_n) \quad \text{and} \quad \|z_n\|_1 \geq \frac{n}{2}
\]

or

\[
\hat{y}_{1/3,2/3}(z_n) \leq 1/2 \leq \hat{y}_{2/3,3/4}(z_n) \quad \text{and} \quad \|z_n\|_1 < \frac{n}{2}
\]

Call the set of such data sets \( Z_{Buy,Sell} \).

Since the state space is binary, each empirical measure \( \hat{Q}(z_n) \) can be identified with the probability \( \hat{p} \) assigned to the state \( y = 1 \). The KL-distance between \( \hat{Q}(z_n) \) and the actual signal-generating distribution \( Q \) is monotonically increasing in \(|\hat{p} - 3/4|\). Thus,

\[
Q^*_{Buy,Sell} = \arg\min_{\hat{Q} \in \{\hat{Q}(z) : z \in Z_{Buy,Sell}\}} D_{KL}(\hat{Q} \| Q)
\]

is the empirical distribution associated with data sets \( z_n \) that minimize

\[
\left| \frac{\|z_n\|_1}{n} - \frac{3}{4} \right|
\]

subject to the constraint that

\[
\hat{y}_{1/3,2/3}(z_n) \leq 1/2 \leq \hat{y}_{2/3,3/4}(z_n)
\]

Notice that \( \hat{y}_{2/3,3/4}(z_n) > 1/2 \) for every \( z_n \) satisfying \( \|z_n\|_1/n > 1/2 \). Therefore the binding constraint is \( \hat{y}_{1/3,2/3}(z_n) \leq 1/2 \). Using (24), this reduces to \( m \leq n/2 \). So we have \( D(Q^*_{Buy,Sell} \| Q) \geq D((1/2,1/2),(3/4,1/4)) \approx 0.14 \). Directly apply Corollary 2 and we are done.

D Supplementary Material to Section 8

D.1 Proof of Proposition 7

The argument is for player 1; the case for player 2 follows identically. For every dataset \( z_n = \{(x_k, y_k)\}_{k=1}^n \), define

\[
Y(z_n) = \{ \hat{f}(x^*) : \hat{f}(x_k) = y_k \quad \forall k = 1, \ldots, n \}
\]

I will show that investment is rationalizable for all types with common certainty in \( Y(z_n) \) if and only if \( Y(z_n) = \{1\} \). Suppose towards contradiction that \( Y(z_n) \neq \{1\} \) so that \( 0 \in Y(z_n) \); then, investment is not rationalizable for player 1 with common certainty in \( y = 0 \). In the other direction, suppose \( Y(z_n) = \{1\} \). Then, common certainty in \( Y(z_n) \) is identical to common certainty that \( y = 1 \), so trivially investment is rationalizable.

Now, observe that \( Y(z_n) = \{1\} \) if and only if every rectangular classification rule \( \hat{f} \) that exactly fits the data predicts \( \hat{f}(x^*) = 1 \). We can reduce this problem to whether the smallest hyper-rectangle that contains every observed vector \( x_k \) also contains \( x^* \). Specifically, the probability \( p^R_n(i, Invest) \) is equal to the probability that on every dimension \( k \),

\[
\exists \text{ observations } (x^i, 1) \text{ and } (x^j, 1) \text{ such that } x^i_k < x^* k \text{ and } x^j_k > x^* k, \quad (25)
\]

that is, a “successful” observation lies on either side of \( x^* \) in dimension \( k \).
Fix an arbitrary \( k \in \{1, \ldots, p^*\} \). Then, (25) is satisfied on dimension \( k \) only if some \( x^i \) satisfying \( x^k_i \in \left[ -c^*, x^*_k \right] \), and also some \( x^j \) satisfying \( x^k_j \in \left( x^*_k, c^* \right] \), are sampled. Since by assumption \( x^*_k \in \left(-c^*, c^*\right) \), the probability that this occurs is

\[
1 - \left[ \left( \frac{2c - c' - x^*_k}{2c} \right)^n + \left( \frac{2c - c' + x^*_k}{2c} \right)^n - \left( \frac{c - c'}{2c} \right)^n \right] = q.
\]

Suppose now that \( k \in \{p^* + 1, \ldots, p\} \). Then, (25) is satisfied on dimension \( k \) only if some \( x^i \) satisfying \( x^k_i < x^*_k \) is sampled, and additionally some \( x^j \) satisfying \( x^k_j > x^*_k \) is sampled. The probability that this occurs is

\[
1 - \left( \frac{c - x^*_k}{2c} \right)^n - \left( \frac{x^*_k + c}{2c} \right)^n = r.
\]

Now, observe that realizations of characteristics are independent across dimensions. So the probability that (25) is satisfied on every dimension is

\[
p^R_{\ln}(i, \text{Invest}) = q^{p^*} r^{p - p^*}.
\]

Since \( r < 1 \), \( p^R_{\ln}(i, \text{Invest}) \) is strictly and monotonically decreasing in \( p \), as desired.

### References


